An examination of the ARX as a residual generator for damage detection

Dionisio Bernal a, Daniele Zonta b, Matteo Pozzi b,
aNortheastern University, Civil Engineering Department, Center for Digital Signal Processing, Boston, 02115
bDepartment of Mechanical and Structural Engineering, University of Trento, via Mesiano 77, 38050 Trento, Italy

ABSTRACT

Residuals that capture the difference between anticipated behavior and actual observations are often used to identify damage. Wanting to control the influence of unmeasured disturbances and noise in residuals, it is common to generate reference signals using feedback from measured outputs. Since there is much flexibility in the gains a wide range of models that react differently to changes are possible. This paper examines two questions: 1) how damage residuals generated by different closed loop models relate to each other and 2) how to rank the expected efficiency of alternative models. On the first question examination shows that the residuals from any model can be viewed as sums of filtered open loop residuals where the filter coefficients depend on the model structure but not on the damage. On the second item a general procedure based on Bayesian decision-making is proposed to quantify the economical benefit in adopting a specific autoregressive model.

Keywords: Open and closed loop predictor models, Residuals analysis, Damage assessment, Bayesian decision making.

1. INTRODUCTION

Finding states that need attention to maintain safety and reliability is the first and most important objective of any health monitoring scheme. The detection task consists in making a binary decision – the data indicates that either the structural or mechanical system is unchanged, or that an undesirable change has taken place. As is well known, the difficulties in robustly detecting damage are connected to uncertainties in the healthy state model, the difficulties in measuring all excitation sources and the inevitable noise associated with the measuring process. Quantitative methods for fault detection can be divided into two groups: 1) parameter estimation techniques and 2) residual based approaches. In the parameter estimation approach the measured data is used to update a model and the parameter fluctuations are used to decide on the existence of significant changes. Residual techniques, instead, establish a model at the outset and code the damage information as discrepancies between measurements and model predictions. Comparing the parameter estimation and residual generation approaches, it may appear that parameter estimation is superior, in the sense that it gives more direct information on the changes; but this is only true if the model parameters have physical meaning and if the resulting problem is well conditioned, requirements that are often difficult to satisfy in real structures.

Since residuals are differences between measurements and reference signals, variations in residual generation schemes correspond to different ways to compute the references. The basic approach to reference signal generation subjects a model that represents the healthy condition to the measured inputs. In this paper we refer to residuals computed this way as open loop (OL) residuals. Wishing to exert some control over the part of the residuals that arise from sources that are extraneous to damage it is common to establish the reference signals using a closed loop structure. The classical scheme closes the loop using the Kalman gain and takes the residuals as the innovation process. The earliest discussion of this strategy known to the writers is by Mehra and Peschon and early implementations can be found in Wilky and Jones and Belkoura. A recent application in mechanics, in the context of structural health monitoring, is presented by Fritzjen and Megelkamp. Other residual generation schemes have used the deviation from orthogonality between matrices that can be computed from the data. In these cases “the model” is the matrix computed for the healthy condition and the residuals are the values obtained when the product that is anticipated to be zero (for ideal conditions) is performed. Examples of this general approach include the many contributions by Basseville and the recent work presented by Yan and Golinval.
This paper focuses on the ARX structure\(^{8,9}\) and addresses the following questions: 1) how do the residuals obtained from different structures relate to each other and 2) how to rank the efficiency of alternative models for the practical purpose of identifying damage and using this information in the rational management of a structure. The paper opens with a discussion on the input-output alternatives for finite dimensional time invariant linear systems and follows with the intended contributions. The analytical part of the paper is followed by a numerical section which adds to the discussion in the context of a particular example.

2. ON THE RELATION BETWEEN DIFFERENT TYPES OF RESIDUALS

2.1 Open and closed loop system equations

To examine the relationship between different ARX residual generators it is convenient to begin by reviewing the issue of input-output maps. As is well known, for a finite dimensional system that displays linear behavior the input-output map can be expressed in non-recursive or recursive form as

\begin{align}
    y(k) &= \sum_{i=0}^{\infty} h_i \cdot u(k-i) \\
    y(k) &= \sum_{i=1}^{n_a} \alpha_i \cdot y(k-i) + \sum_{j=0}^{n_b} \beta_j \cdot u(k-j)
\end{align}

where \(y(k)\) and \(u(k)\) are the \(m\) and \(r\) dimensional output and input vectors respectively, \(h, \alpha\) and \(\beta\) are matrices of appropriate dimensions and \(n_a\) and \(n_b\) are integers. Eq. 1 is the discrete time version of the continuous time convolution input-output relation, often referred to as the weighting sequence description and, in signal processing literature, in scalar form, as the finite impulse response filter or FIR\(^{10}\).

The map in Eq. 2 is known as the ARX model, as a dead beat observer in the state space framework\(^{8}\) and (again in scalar form) as the infinite impulse response filter or IIR in digital signal processing\(^{10}\). The orders \(n_a\) and \(n_b\) have lower bounds that depend on the order of the system and the number of measurements, but are not bounded from above. The flexibility in \(n_a\) and \(n_b\) is reflected in the fact that the matrices \(\alpha\) and \(\beta\) are not unique. Particular forms that are useful in some applications are obtained by fixing some of the matrices (or individual entries in the matrices). For example, the M-step ahead predictor, widely used in control applications is obtained by taking \(\alpha = 0\) for \(i = 1, 2,.. M-1\).

Once the model parameters are estimated from the measurements\(^{11}\) Eqs. 1 or 2 can be used – given a new data set – to predict the output. The OL predictions are those obtained from Eq. 1 or, equivalently, from Eq. 2 when the response terms on the right hand side are taken as feedback from the left hand side. When the measured output is used on the \(rhs\) of Eq. 2 the predictions depend (except in the ideal case where the data fits the model exactly) on the specific model structure. It is opportune to note that in estimating \(\alpha\) and \(\beta\) it is possible to contemplate the existence of unmeasured disturbances and noise in the measurements. For example, when the disturbances and the noise are white a Kalman estimator\(^{11,12}\) provides unbiased estimates of the required parameters.

Fig.1 reports the whole scheme of residual-based damage detection. After the model is tuned by a characterization test, it is employed to predict the structural response during any inspection test. The difference between the actual and the predicted response is the residuals, which is measured to quantify the distance between the original and the actual structure. This norm is employed, in turn, in the decision making process related to the management of the structure.
2.2 Decoupling damage from model structure

In this section we examine the first question that motivates the paper, namely: how do residuals from various closed loop ARX structures relate to each other. To begin we assume that an input output map has been obtained for the healthy state and that a set of input-output measurements has been collected subsequently. From Eq. 2 and the definition of residuals one can write

\[ \text{res}(k) = y_m^d(k) - \sum_{i=1}^{n} a_i \cdot y_m^d(k-i) - \sum_{j=0}^{n} \beta_j \cdot u(k-j) \]  

where the subscript \( m \) in \( y_m^d \) stands for measurements and the superscript \( d \) suggests that these measurements are from a (potentially) damaged state. To eliminate dependence on the measured inputs from Eq. 3 we write the open loop prediction of the output for the reference state in recursive form

\[ y_p^u(k) = \sum_{i=1}^{n} a_i \cdot y_p^u(k-i) + \sum_{j=0}^{n} \beta_j \cdot u(k-j) \]  

where the subscript \( p \) stands for predicted and the superscript \( u \) for undamaged, and solve for the exogenous part of the model. Substituting the result into Eq. 3 gives

\[ \text{res}(k) = y_m^d(k) - \sum_{i=1}^{n} a_i \cdot y_m^d(k-i) - y_p^u(k) + \sum_{i=1}^{n} a_i \cdot y_p^u(k-i) \]  

Designating the nominal open loop residuals as

\[ \epsilon_{nom}(k) = y_m^d(k) - y_p^u(k) \]  

Eq. 5 becomes

---

Fig. 1. Scheme of the residuals computation, analysis and implementation into the decision making.
\[ \text{res}(k) = \epsilon_{nom}(k) - \sum_{i=1}^{n} a_i \cdot \epsilon_{nom}(k - i) \]  

or, in more compact form

\[ \text{res}(k) = -\sum_{i=0}^{n} a_i \cdot \epsilon_{nom}(k - i) \]  

where \( a_0 \) is the negative of the identity.

Eq. 8 shows that the output residuals for any ARX model are sums of filtered versions of the nominal open loop residuals. Since the filtering functions (the \( a \)) depend on the model structure, but not on the damage, the selection of a given structure is tantamount to the specification of a set of (causal) filters which operate on \( \epsilon_{nom} \) to provide the residuals at each channel.

An alternative formulation relies on the hypothesis of an ability to drive the original structure in such a way that it gives out the actual output: \( y^d_m : \) vector \( u^* \) describes the equivalent forces which feed the original structure. If we succeed in following the actual output, the following relation holds:

\[ y^d_m(k) = \sum_{i=1}^{n} a_i \cdot y^d_m(k - i) + \sum_{j=0}^{n} \beta_j \cdot u^*(k - j) \]  

Let us define the nominal forcing residuals as the difference between actual and equivalent forces:

\[ \Delta u(k) = u(k) - u^*(k) \]  

By substituting Eqs. 9 and 10 into Eq. 3, we can express the residuals as a function of the nominal forcing residuals:

\[ \text{res}(k) = \sum_{j=0}^{n} \beta_j \cdot \Delta u(k - j) \]  

Eqs. 8 and 11 are alternative formulations of the residuals: the former employs correlation functions \( a \), the latter functions \( \beta \). However it is worth noting that Eq. 8 is of general application, while Eq. 11 is not: in general it is not possible to find the equivalent forces \( u^* \), because the number of actuators could be too small compared with the number of sensors. Fig. 2 reports the schemes of the physical meaning of both the nominal OL residuals and of nominal forcing residuals, as well as the relation between them.

This paragraph proves that the residuals generated by whatever ARX model are deterministically related to the same quantity, e.g. the nominal OL residuals, by Eq. 8. Furthermore this transformation is damage independent, as it based only on correlation functions \( a \). Hence, one can draw the conclusion that the damage-related information encoded into the residuals generated by whatever model is the same and that consequently all models are equivalent for the sake of damage detection. However this conclusion holds insofar as no noise or uncertainty is included in the process. In this ideal case, only two outcomes are possible: either the actual system dynamic during the inspection is equal to the original, and thus the residuals are null, or the system has changed and residuals are not null.
2.3 Noise related effects

In a real application disturbances are unavoidable. In these circumstances, residuals are not zero even in the undamaged scenario, and they have to be processed to gain information on the system state. The nominal OL residuals in the presence of noise can be expressed as:

\[ \varepsilon_{nom}(t) = \varepsilon_{dam}(t) + z(t) + v(t) \]  (12)

where \( \varepsilon_{dam}(t) \) is the damage-related residual (which is null if the structure is undamaged), \( z(t) \) is output of the reference system when subjected to the unmeasured disturbances only and \( v(t) \) is the output noise.

While the nominal OL residuals are univocally defined, the residuals related to a specific ARX model depend on the corresponding correlation function \( \alpha \), according to Eq. 8. Thus the model acts as a filter on the nominal residuals and, depending on the model, this filter can highlight either the damage-related or the spurious components. It is therefore mandatory to investigate the behavior of different models, and to define a procedure for discriminating the efficiency of a model for damage detection: this will be done in section 3.

2.4 Residuals norm

The amplitude of the residuals can be measured by a single scalar norm. Here we follow the same approach proposed by Bernal and Hernandez\[13\], where essentially the metric selected is the Euclidean norm of the residual history during the observation time. In this case the structure is linear, so it appears reasonable to normalize this quantity to the Euclidean norm of the response in such a way as to operate with an index somehow independent of the response amplitude. In the equation we have:

\[ X = \sqrt{\sum_{c=1}^{m} \int res_{c}^{2}(t)dt \over \sqrt{\sum_{c=1}^{m} \int y_{c}^{2}(t)dt}} \]  (13)

where index \( c \) indicates the \( c \)-th channel and \( m \) is the number of channels.

Fig. 2. Schematic definition of the nominal OL residuals and of nominal forcing residuals.
3. PROBABILISTIC EVALUATION OF THE EFFECTIVENESS OF DIFFERENT MODELS

In this section we investigate a rational approach to ranking the ability of different autoregressive models, defined by different values of $\alpha$ and $\beta$ in Eq. 2, to detect damage. At first sight, this problem may seem trivial, believing that the best model is that exhibiting the higher residual when the structure is damaged. This common, but erroneous, conviction arises from the observation that, in ideal conditions, the residual is null for intact structures, and not null for damaged ones: therefore, the impression is that the greater the residual, the more evident the damage. In reality, if we see the problem from a deterministic point of view, any autoregressive model or metric is equally good at detecting damage, because the damaged state, yielding a non zero residual, is always perfectly distinguishable from the undamaged one.

In real life we have to deal with uncertainties stemming from modeling errors, instrumental noise and unknown inputs, quantities that here we will generically indicate as noise. As Eq. 12 points out, the issue is that the different autoregressive models filter in different ways not only the damage feature $\varepsilon_{\text{dam}}(t)$, but also disturbances $v(t)$ and $z(t)$: due to noise, the residual might be significantly greater than zero even in the undamaged case. A model is effective when its response in the damaged situation is not likely to be confused with that in the undamaged situation. Because the response of the model depends on noise this problem must necessarily be investigated in a probabilistic framework.

The authors addressed this problem in a contribution recently presented at the IMAC\textsuperscript{[14]}: in that paper, we used the empirical Probability Density Functions (PDF) of the residual norm $\chi$ produced in the damaged and intact situation as a qualitative tool to evaluate the ability of the model to discriminate the two conditions. Qualitatively, one can easily understand that when the two distributions are well separated, with essentially no overlap, the damage can easily be detected. On the contrary, the decision is very difficult when the two distributions tend to be similar.

3.1 Problem formulation

In the remaining part of this section, we want to formalize this qualitative observation with a quantitative criterion. For instance, a possible approach, reflecting the common sense of those in damage detection, is to state that the best model is that exhibiting the higher residual when the structure is damaged. On the contrary, the decision is very difficult when the two distributions tend to be similar.

In principle, we can state the problem from a very general point of view, defining manifold scenarios and different possible options in structure management. Here it is useful to offer the reader an idea of how the general approach works in a simple and specific case. Assume for example, that we are monitoring a bridge that can be in only one of two possible options in structure management. Here it is useful to offer the reader an idea of how the general approach works in a simple and specific case. Assume for example, that we are monitoring a bridge that can be in only one of two discrete states, damaged (D) or undamaged (U), which represent a set of mutually exclusive and exhaustive possibilities:

$$\text{prob}(D) + \text{prob}(U) = 1$$

The bridge is monitored with a number of instruments and the data collected by these instruments are processed using an autoregressive algorithm $M$. The response delivered to the manager of the bridge is, in essence, the residual norm $\chi$. Based on this outcome the manager can take some actions. Qualitatively, if the response of the monitoring does not cause concern, he can decide to do nothing. Conversely, if the value of $\chi$ is a source of concern, he will be prompted to take some action. Again, keeping the problem simple, we assume that the only option, alternative to do nothing, is to carry out further inspection on the bridge, to better understand the nature of damage. Any action selected by the manager may involve a cost, and in general this cost depends on the effects of the decision. For instance, doing nothing has no cost if the structure is actually safe, while underestimating the damage condition will result in a future economic cost of...
repairing and possibly in an indirect social cost when this situation jeopardizes user safety. Without going into detail, we can suppose that both direct and indirect costs can be monetized and summarized as an overall undershooting cost \( C_{US} \). The cost of inspection includes primarily the direct economic cost inherent in practical execution of the inspection; in addition, during the inspection, the bridge has to be closed to traffic and this involves an indirect cost to the user. So we can in general state that if an inspection is undertaken, this will involve a cost \( C_I \); and this cost is independent of the actual state, damaged or undamaged, of the structure. These concepts can be formally summarized in the following cost-per-action table:

<table>
<thead>
<tr>
<th>Action</th>
<th>System State</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Undamaged (U)</td>
<td>Damaged (D)</td>
</tr>
<tr>
<td>Do nothing (N)</td>
<td>0</td>
<td>( C_{US} )</td>
</tr>
<tr>
<td>Inspection (I)</td>
<td>( C_I )</td>
<td>( C_I )</td>
</tr>
</tbody>
</table>

Table I: Costs per action and state

Based on this table, the manager estimates the cost involved in any action. While the cost of inspection is identically equal to \( C_I \), regardless of the state of the bridge, the cost of the do-nothing scenario depends on the manager’s estimation of the probability of being in the damaged state. Assume, as a first step, that the manager has no knowledge of the outcome of monitoring; then, he will quantify the cost of the do-nothing option as:

\[
C_N = C_{US} \text{prob}(D)
\]

(16)

where \( \text{prob}(D) \) is the probability \textit{a priori} of structural damage, evaluated based on his prior knowledge or experience. At this point we must define the principle driving the manager’s decision. If the manager behaves rationally (not necessarily the most common case) we can assume that he will decide with the objective of minimizing the expected cost. More specifically, he will carry out the inspection when \( C_N > C_I \) and he will do nothing when \( C_I > C_N \). In conclusion the loss \( C^* \) estimated by the manager in the prior situation is:

\[
C^* = \min(C_N, C_I)
\]

(17)

As a second step, let us analyze how the knowledge of \( \chi \), changes the manager’s decision, and consequently the expected loss associated to this decision. The decision-maker knows the distributions PDF(\( \chi \mid D \)) and PDF(\( \chi \mid U \)) of \( \chi \) for model \( M \) in the damaged and intact cases. For those familiar with Bayesian methods these distributions are also known as the \textit{evidence of damage} and \textit{evidence of no-damage}. Using his/her prior judgment the decision-maker can also predict the distribution of \( \chi \), before this data is available, by marginalizing on the system states:

\[
\text{PDF}(\chi) = \text{PDF}(\chi \mid D) \cdot \text{prob}(D) + \text{PDF}(\chi \mid U) \cdot \text{prob}(U)
\]

(18)

This latter PDF is usually referred to as \textit{model evidence} in Bayesian theory. Now, when the monitoring result \( \chi \) is available to the manager, he can update his estimate of the probability of damage using Bayes’ formula:

\[
\text{prob}(D \mid \chi) = \frac{\text{PDF}(\chi \mid D) \cdot \text{prob}(D)}{\text{PDF}(\chi)}
\]

(19)

where \( \text{prob}(D \mid \chi) \) is the posterior probability of damage estimated using model \( M \). The new \textit{(a posteriori)} estimates of the losses associated to actions do-nothing or inspection are:

\[
M C_{N\prime} = C_{US} \cdot \text{prob}(D \mid \chi) \quad M C_I = C_I
\]

(20)

Once again, if the manager’s decision is driven by an economic principle, he/she will choose the least expensive option, thus the loss associated to the decision, in the posterior condition, is:

\[
M C^*(\chi) = \min(M C_{N\prime}, M C_I) = \min(C_N, C_{US} \cdot \text{prob}(D \mid \chi))
\]

(22)

This point is very important and deserves the reader’s attention: the manager will decide for an inspection not when the damage detection shows that damage is more likely than non-damage, but when the loss expected for doing-nothing is greater that the cost of an inspection \( C_I \).
3.2 Norm of effectiveness of a model

Eq. 22 refers to the case when a specific outcome $\chi$ is available. We now want to calculate a priori the loss expected following application of model $M$. To do so we have to marginalize the loss expressed by Eq. 22 using the model evidence:

$$M C^* = \int_{0}^{M C^* (\chi)} M PDF(\chi) \, d\chi = \int_{0}^{\min\{C_{US} PDF(\chi | D) \text{prob}(D) ; C_{F} PDF(\chi)\}} d\chi \tag{23}$$

This quantity, labeled model cost, encodes the expected cost of a decision process based on the outcome of model $M$, and can be seen as a metric for evaluating the effectiveness of the model. As it is evident from Eq. 23, the model cost depends not only on the PDFs of $\chi$, but also on the scenarios’ costs and on the prior knowledge of the system state. To compare the effectiveness of different models it is further convenient to introduce the non-dimensional norm:

$$\rho_M = 1 - \frac{M C^*}{C^*} \tag{24}$$

This norm represents the percentage of economic saving associated to the specific model, varying in the present case from zero when the model is economically useless, to the upper limit of 100%, when the application of the model virtually allows elimination of any possible posterior cost.

4. NUMERICAL SIMULATION

To illustrate how the ranking procedure works, let us examine a numerical example. Fig.3 shows the scheme of a cable stayed bridge, with a single-span of 120m, supported by 3 cables placed at 30m pitch. The deck is continuous at the cable anchorages and simply supported at the ends. The damage simulated is the partial release of the upper cable, with damage to one of its two strands, so that the stiffness is reduced to 50% (as also the load bearing capacity).

The bridge response is simulated by a toy FEM, built with beam elements, with 14 degrees of freedom. The bridge is instrumented by 4 vertical sensors, labelled $s_1$…$s_4$, placed at the mid-points between the cable anchorages (at points 1, 3, 5, 7 on the FEM), and with a single actuator, at the deck mid-span (at point 4). The damage affects the stiffness matrix and thus the bridge response. The test is assumed in the following form. The actuator is driven by a 10 seconds random force, acquired with a sample frequency of 50Hz, normally distributed with zero mean and unit standard deviation. The response of the sensors is recorded with the same frequency. Uncertainties are modeled by two main sources: each sensor response is superimposed by white noise with 5% noise to signal ratio, and an unknown white noise force with 5% standard deviation is applied to each of the 8 points: thus, both input and output noise is considered.

![Fig.3. Scheme of the cable-stayed bridge taken as case study.](image-url)
We compute residuals for two specific models. The first is the standard one-step-ahead-predictor ARX model, and the other is a model obtained by taking the diagonal of $\alpha$ equal to zero (except for $\alpha_0$, which as noted previously, is the negative of the identity): in other words, the prediction at a given channel is obtained without using the past of this channel. This model, originally introduced by Bernal and Hernandez\cite{13} in a study on earthquake damage detection in multistory buildings, is designated as the Partial Observer Model or POM. The qualitative argument suggesting adoption of a POM rather than an ARX model is that the former puts less constraint on the system dynamic and hence better allows damage related effects to become apparent. For both models, we selected $n=40$.

![Fig.4: First three mode shapes of the bridge, in the undamaged and in the damaged conditions.](image1)

![Fig.5. Time histories and Fourier spectra of residuals resulting from the ARX (left) and POM (right) models.](image2)
Fig. 5 reports two examples of residual time history, both for the ARX and the POM models, in the damaged state, both in the time and in the frequency domains. As is apparent, the residuals produced by the POM model are larger: this is because the prediction on each channel relies only on measurements taken on other channels and, as a consequence, the difference to the actual response is larger. However, since this phenomenon occurs even in the undamaged case, no direct conclusion on the model efficiency can be drawn from it.

For comparison of the two models, 100 Monte Carlo simulations are launched both in the undamaged and in the damaged cases, and the response time histories are examined both with the ARX and with the POM model. Thus 100 samples of residual norm $\chi$ are derived for each model and for each system state.

The two model likelihoods are derived by fitting the Monte Carlo samples to log-normal distributions. In Fig. 5, blue crosses show the frequency outcomes of the simulations in damaged condition, while the continuous line shows the fitting curve $M^{PDF}(\chi|U)$; the red crosses and line refer to the damage condition. The upper graph is related to the ARX model, while the lower one to the POM. As a result of the fitting procedure, Fig. 6 shows the model likelihoods and Table 2 reports the parameters of the four fitted distributions.

We suppose that the damage is considered a priori quite unlikely, so that $\text{prob}(D)=10\%$, thus $\text{prob}(U)=90\%$. The cost of an inspection is quantified in $C_I=30\text{k€}$, that of an undershooting is estimated in $C_{US}=200\text{k€}$. According to Eq. 16, before carrying out the test, the expected loss related to the do-nothing option is $C_N=20\text{k€}$: hence, this is the best prior option. The upper row of Fig. 7 reports the fitted curves for model likelihoods reported in the previous picture, as well as the corresponding model evidence (black dashed lines), as functions of the test output $\chi$. The central row shows the updated probability of the damaged and the undamaged states, while the lower row reports the loss related to options $N$ and $I$, the optimum loss $M^{C^*(\chi)}$ and the expected model loss $C^*(\chi)$.

Table 2. Statistical parameters of the four log-normal likelihood distributions.

<table>
<thead>
<tr>
<th>distribution</th>
<th>model</th>
<th>$\mu$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{(ARX)}PDF(\chi</td>
<td>U)$</td>
<td>ARX</td>
<td>0.029</td>
</tr>
<tr>
<td>$^{(ARX)}PDF(\chi</td>
<td>D)$</td>
<td>ARX</td>
<td>0.031</td>
</tr>
<tr>
<td>$^{(POM)}PDF(\chi</td>
<td>U)$</td>
<td>POM</td>
<td>1.4</td>
</tr>
<tr>
<td>$^{(POM)}PDF(\chi</td>
<td>D)$</td>
<td>POM</td>
<td>2.0</td>
</tr>
</tbody>
</table>
Fig. 7: Probabilities and expected losses for the case study analysis.
As a result of applying Eqs. 20-24, the expected loss is 18.2k€ using the ARX model and 12.1k€ for the POM. The corresponding saving ratios are $\rho_{\text{ARX}} = 9\%$ and $\rho_{\text{POM}} = 39\%$, respectively. We conclude that, for this particular problem and the specific decision criterion adopted, the POM model is more effective than the classical ARX.

5. CONCLUSIONS

The analysis presented shows that use of an ARX structure to generate residuals for damage detection is relevant to the specification of the set of filters that act on the Open Loop ones. Since the coefficients of these filters do not depend on the damage (i.e. they are not adaptive) any ARX residuals (including OL and POM) are equivalent, in the sense that one can go from one to another without knowledge of the damage. However, this equivalency does not imply that the PDFs of the residual metric in the reference and the damaged state display the same contrast. A rational comparison between alternative models has to investigate their behavior in a noisy and realistic condition. The empirical Probability Density Function (PDF) of the residual norm produced in the damaged and intact situation is a qualitative tool to evaluate the ability of the model to discriminate the two conditions. The quantitative formalization of this concept requires recasting the problem within a probabilistic decision-making framework, whereby the most effective model is that allowing minimization of the posterior economic cost. It is crucial to remember that the decision-maker will decide for an action not when damage detection shows that the damage is more likely than the non-damage, but when the loss expected for doing-nothing is greater than the cost of that action.

REFERENCES