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RESPONSE SPECTRUM COMPATIBLE PSD FOR HIGH-FREQUENCY RANGE

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ABSTRACT

A response-spectrum-compatible power spectral density (PSD) is used in design of nuclear power plants to circumvent costly time history analysis. Several methods are available for identifying the compatible PSD. However, investigators have recently noted that compatibility between the response spectrum and the corresponding PSD may be lacking in the high frequency range. In this paper we examine three methods for generating a PSD corresponding to a response spectrum that is compatible for the entire range of frequencies. We also discuss conditions on the response spectrum shape to assure this level of compatibility.

INTRODUCTION

Seismic design requirements for nuclear power plants are usually specified in terms of design response spectra. It is a standard practice in the nuclear industry to generate suites of spectrum-compatible time histories and perform time-consuming response history analysis to satisfy code requirements or to obtain probabilistic estimates of the response. An alternative to this approach is to generate a response-spectrum-compatible power spectral density (PSD) and perform random vibration analysis instead of time-history analysis. Aside from the computational savings, the random vibration approach does not suffer from the arbitrariness of the selected time histories and the difficult task of selecting a sample size.

A number of methods have been proposed for generating the response-spectrum-compatible PSD. However, recently some investigators have noted that the compatibility between the response spectrum and the corresponding PSD may be lacking in the high frequencies (higher than 10Hz) range (Ostadan 2011). Specifically, when the power spectral density is converted back into a response spectrum, the resulting response spectrum does not match the target response spectrum in the high frequency range.

This paper investigates the above problem and offers three approaches that provide consistent compatibility for all frequency ranges. The paper also examines the requirements on the shape of a response spectrum that must be satisfied in order to generate a compatible PSD.

STATEMENT OF THE PROBLEM

Several methods are available in the literature to generate PSDs compatible with a given response spectrum. The basic identity used to generate the compatible PSD is a relation linking the response spectrum to the variance of the modal response:

$$\frac{[A(\omega)]^2}{[p(\omega)]^2} = \sigma^2(\omega) = \int_0^\infty |H(\omega,\gamma)|^2 \phi(\gamma) d\gamma \tag{1}$$

In the above equation, $A(\omega)$ is the pseudo-acceleration response spectrum, $p(\omega)$ is the corresponding peak factor, which is typically a slowly varying function of frequency, $\sigma^2(\omega)$ is the variance of the pseudo-acceleration response of an oscillator with angular frequency ω, $\phi(\gamma)$ is the one-sided compatible PSD, and $H(\omega,\gamma)$ is the Frequency Response Function (FRF) for pseudo acceleration response of the oscillator forced at frequency γ and is given by

$$H(\omega,\gamma) = \frac{\omega^2}{\omega^2-\gamma^2+2\mathrm{i}\zeta\omega\gamma} \tag{2}$$

in which ζ denotes the damping ratio of the oscillator and $\mathrm{i} = \sqrt{-1}$ is the imaginary unit. Eq. 1 is based on the assumption of stationary input and output processes. The first identity in Eq. 1 derives from the definition of the peak factor as the ratio between the maximum and the standard deviation of a stationary process, while the second one is the fundamental relation between the mean-square stationary response and the input PSD (Lutes and Sarkani, 2004).

We face an inverse problem: Given the response spectrum $A(\omega)$, we aim at identifying the corresponding PSD $\phi(\omega)$, using Eq.1. We require that the function $\phi(\omega)$ be real and non-negative, since it represents the PSD of a real process. We note that there is no guarantee that Eq.1 admits solutions for an arbitrary response spectrum shape. If it does not admit a solution, then it means that the proposed response spectrum is not compatible with the fundamental rules of random vibration theory that are behind Eq.1.

For a given variance function $\sigma^2(\omega)$, Eq.1 has the form of an inhomogeneous Fredholm integral equation of the first kind, where $|H(\omega,\gamma)|^2$ is the kernel function. By changing variables, taking the logarithm of the frequency, Eq.1 can be re-shaped as a convolution integral, which can be solved by means of the direct and inverse Fourier transform (Polyanin and Manzhirov, 1998). However, when the function $\sigma^2(\omega)$ admits no exact solutions for a non-negative function $\phi(\omega)$, it is convenient to apply approximate methods. The following section provides a review of these methods.

INCREMENTAL APPROACH BY VANMARKE

An approximate solution for the compatible PSD can be derived by the approach proposed by Vanmarke (1977), which is based on an approximation of the kernel function in Eq.1. We start by adding and subtracting, in Eq.1, the value of the PSD at the modal frequency ω:

$$\sigma^2(\omega) = \int_0^\infty |H(\omega,\gamma)|^2[\phi(\gamma) - \phi(\omega) + \phi(\omega)]d\gamma \tag{3}$$

The above integral is now evaluated in two parts:

$$\begin{cases} \int_0^\infty |H(\omega,\gamma)|^2 \phi(\omega) d\gamma = \frac{\pi\omega}{4\zeta}\phi(\omega) \\ \int_0^\infty |H(\omega,\gamma)|^2[\phi(\gamma) - \phi(\omega)]d\gamma \cong \int_0^\omega \phi(\gamma)d\gamma - \omega\phi(\omega) \end{cases} \tag{4a,b}$$

The first identity is the well-known result for response to white noise (Lutes and Sarkani, 2004). The second approximation is based on the following argument: For a small damping ratio ζ, $|H(\omega,\gamma)|^2$ is approximately equal to 1 for frequency γ from zero up to a frequency slightly smaller then ω, while it is nearly zero for frequencies above a frequency slightly higher than ω. Around the modal frequency ω, due to resonance, the function reaches a peak value of $\zeta^2/4$. However, the quantity in the square brackets in Eq.4b is nil at $\gamma = \omega$. Thus, provided the PSD is sufficiently smooth, we can argue that the contribution of the resonance effect in the second integral is negligible. Of course, the resonance is fundamental in the integral in Eq.4a. Substituting Eq. 4 into 3, we obtain

$$\sigma^2(\omega) = k\omega\phi(\omega) + \int_0^{\omega} \phi(\gamma)d\gamma \tag{5}$$

where $k = (\pi - 4\zeta)/4\zeta$. Re-arranging the terms and replacing the variance by the identity in Eq.1, we end up with the following approximation:

$$\phi(\omega) = \frac{1}{k\omega}\left[\left(\frac{A(\omega)}{p(\omega)}\right)^2 - \int_0^{\omega} \phi(\gamma)d\gamma\right] \tag{6}$$

Eq.6 is an incremental formula to compute $\phi(\omega)$. As a first approximation, the peak factor may be set to a constant value, say $p(\omega) = 3$. We start at $\omega = 0$, imposing the initial conditions $\phi(0) = 0$ and $\phi'(0) = 0$ to assure finite ground velocity. We then proceed to compute $\phi(\omega)$ for increasing frequencies by numerical integration.

TAIL BEHAVIOR

It is well known that the pseudo-acceleration response spectrum must approach the peak ground acceleration, a_g, at high frequencies. It follows that the response variance function must approach the variance of the ground acceleration, $\sigma^2(\omega) = \sigma_g^2 = \int_0^{\infty} \phi(\gamma)d\gamma$ as $\omega \to \infty$. Using this relation in Eq.5, we can write

$$\sigma^2(\omega) = k\omega\phi(\omega) + \sigma_g^2 - \int_{\omega}^{\infty} \phi(\gamma)d\gamma \tag{7}$$

Let ω_m be an anchor frequency in the high-frequency tail. Since the PSD must be integrable to achieve a finite peak ground acceleration, we adopt the power tail form

$$\phi(\omega) \cong \phi(\omega_m)\left(\frac{\omega_m}{\omega}\right)^{1+\alpha}, \quad \omega_m < \omega \tag{8}$$

where $\alpha > 0$. By substituting Eq.8 in Eq.7, we obtain the following tail approximation:

$$\left(\frac{A(\omega)}{p(\omega)}\right)^2 = \sigma^2(\omega) \cong \sigma_g^2 + \left(k - \frac{1}{\alpha}\right)\phi(\omega_m)\frac{\omega_m^{1+\alpha}}{\omega^{\alpha}}, \qquad \omega_m < \omega \tag{9}$$

Rearranging terms and taking the logarithm, we obtain

$$\log\left[\left(\frac{A(\omega)}{p(\omega)}\right)^2 - \sigma_g^2\right] \cong b - \alpha\log(\omega), \qquad \omega_m < \omega \tag{10}$$

with $b = \log(k - 1/\alpha) + \log[\phi(\omega_m)] + (1 + \alpha)\log(\omega_m)$. For a specified response spectrum $A(\omega)$, setting $p(\omega) = 3$, we can calibrate parameters b and α by examining two points in the tail of the spectrum. This helps us determine the tail behavior of the PSD function.

COMPATIBILITY CONDITIONS

Equation 6 imposes certain restrictions on the variance function $\sigma^2(\omega) = [A(\omega)/p(\omega)]^2$ in order to have a compatible PSD. Since the PSD must be non-negative, for any frequency ω_a, the integral $I(\omega_a) = \int_0^{\omega_a} \phi(\gamma)d\gamma$, which is non-decreasing in ω_a, imposes a lower bound on the values of $\sigma^2(\omega)$ for frequencies higher than ω_a. In other words, $A(\omega)$ cannot have too fast a decay with ω. This makes sense since an oscillator with a frequency higher than ω_a is also affected by the input power in the frequency range below ω_a and, therefore, its response cannot be arbitrarily small. This requirement imposes a condition on the shape of the response spectrum. To check if a given response spectrum is admissible, we apply Eq.6. If, for any frequency, the function $\phi(\omega)$ is negative, we consider the response spectrum as inadmissible. This requirement of course relies on the assumptions supporting Eq.6, which are stationary response and small damping ratio ζ.

Equation 10 imposes an additional constraint on the tail of the response spectrum. Namely, for a large ω_m, the value obtained for the parameter α must be positive. If not, the tail of the response spectrum is not decaying sufficiently fast.

A given response spectrum may be inadmissible for a number of practical reasons. For example, in design codes, the response spectrum is usually prescribed in parametric form, with the parameters depending on site and soil conditions. The spectrum shape is usually constructed by processing a set of ground motion records and fitting a simple analytical function to their envelope. There is no attempt to verify that the conditions described in this section are fulfilled. For this reason, we believe response spectra specified in certain codes may be inadmissible.

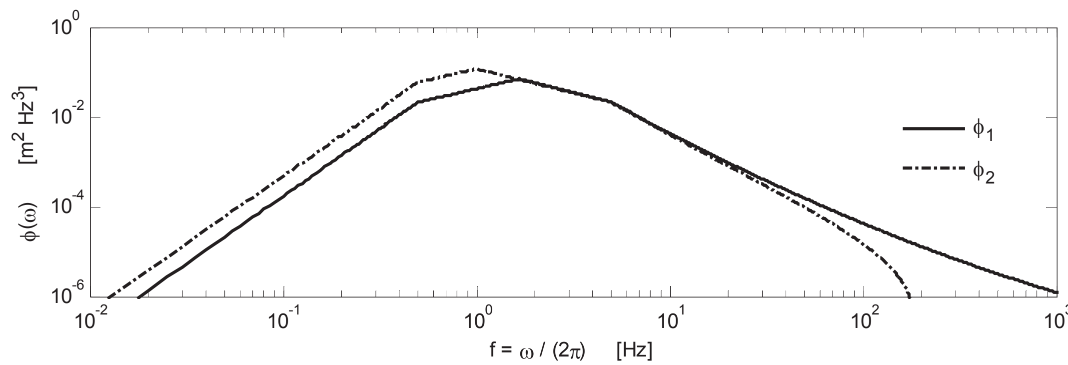

Figure 1. Identified PSDs for two target response spectra: a compatible spectrum (continuous line) and incompatible spectrum (dashed-dotted line).

EXACT SOLUTION FOR SPECIFIC VARIANCE FUNCTIONS

Equation 6 yields closed-form solutions for certain variance functions. Let $\sigma^2(\omega) = \beta\omega^{-n}$ for $\omega_0 \leq \omega$. The corresponding solution of the PSD is:

$$\phi(\omega) = \frac{1}{\omega}\left[-\frac{1}{k}\left(\int_0^{\omega_0}\phi(\gamma)d\gamma + \frac{\beta\omega_0^{-n}}{nk-1}\right)\left(\frac{\omega_0}{\omega}\right)^{\frac{1}{k}} + \frac{n\beta}{nk-1}\omega^{-n}\right] \tag{11}$$

This solution is applicable to positive, negative or zero values of n, the latter case representing a constant variance. The above solution allows constructing exact solutions of Eq.6 for any piecewise variance function, where each piece is defined in the manner described above. In particular, design response spectra in modern building codes are usually defined by piecewise linear combinations of such functions. Assuming a constant peak factor, the corresponding variance function can be expressed in the manner described above.

Figure 1 shows the computed PSDs for two target response spectra. We assume a constant peak factor, $p(\omega) = 3$. Both spectra have PGA $a_g = 4\text{ms}^{-2}$, a pseudo-acceleration spectrum value linearly increasing with the natural period T up to 10ms^{-2} at $T = 0.2\text{s}$, constant up to T_c, proportional to T^{-1} up to 2s, and proportional to T^{-2} above that. PSD function ϕ_1 in Figure 1 is computed assuming $T_c = 0.6\text{s}$, while PSD function ϕ_2 assumes $T_c = 1\text{s}$. In the former case we can find a compatible PSD; in the latter case the identified PSD becomes negative at frequency 194Hz. This indicates that the response spectrum with $T_c = 1\text{s}$ is inadmissible. This happens because the drop in the response spectrum value down to the PGA is too fast for the long constant-acceleration phase of the spectrum.

DER KIUREGHIAN AND NEUENHOFER' APPROACH

An alternative method to identity the response-spectrum-compatible PSD was proposed by Der Kiureghian and Neuenhofer (1992). This is an iterative scheme to solve Eq.1, which does not require the approximation employed in Eq.4. The initial approximation is based on response to white noise, according to which $\phi(\gamma)$ in Eq.1 is replaced by its value at the resonance frequency, $\phi(\omega)$, yielding

$$\sigma^2(\omega) \cong \phi(\omega)\int_0^\infty |H(\omega,\gamma)|^2 d\gamma = \frac{\pi\omega}{4\zeta}\phi(\omega) \tag{12}$$

The corresponding first approximation of the PSD is

$$\phi_{(1)}(\omega) = \frac{4\zeta}{\pi\omega}\sigma^2(\omega) \tag{13}$$

We now can solve the *direct* problem, i.e., obtain the variance function for the given PSD. To do this, we discretize the frequency axis and numerically perform the integration for each ω. After k iterations, we have

$$\sigma_{(k)}^2(\omega) = \int_0^\infty |H(\omega,\gamma)|^2 \phi_{(k)}(\gamma) d\gamma \tag{14}$$

The updated PSD can now be written as

$$\phi_{(k+1)}(\omega) = \frac{\sigma^2(\omega)}{\sigma_{(k)}^2(\omega)}\phi_{(k)}(\omega) \tag{15}$$

The iterations converge when the computed variance function $\sigma_{(k)}^2(\omega)$ from Eq.14 matches the target variance function $\sigma^2(\omega)$.

In principle, this iterative approach may start with a first approximation different from that in Eq.13. However, the white-noise approximation guarantees that the starting trial function is non-negative. Note that all subsequent functions in the sequence are also non-negative, as is evident in Eq.15. Thus, using this method, we never face the inconsistent result of a negative PSD. In the case of using an inadmissible response spectrum, the method converges to a PSD that is not consistent with the target variance for high frequencies. More specifically, if the PSD is used to generate a corresponding response spectrum, the result does not match the target response spectrum in the high frequency range.

NUMERICAL SCHEME BASED ON QUADRATIC PROGRAMMING

Equation 1 can also be solved numerically. We discretize the variance function into vector $\boldsymbol{\sigma}^2$, the PSD into vector $\boldsymbol{\phi}$, and the kernel function into matrix $\mathbf{H}^2$ so that, in the discretized space, Eq.1 reads $\boldsymbol{\sigma}^2 = \mathbf{H}^2\boldsymbol{\phi}$. Depending on the discretization employed, the matrix $\mathbf{H}^2$ may or may not be invertible. In any case, we can characterize the space of the admissible variance functions as the part of the image of matrix $\mathbf{H}^2$ referring to vectors $\boldsymbol{\phi}$ with only non-negative components.

When matrix $\mathbf{H}^2$ is not invertible, we find an optimal vector $\boldsymbol{\phi}$ by minimizing a cost function $C(\boldsymbol{\phi})$ defined as

$$C(\boldsymbol{\phi}) = \|\boldsymbol{\sigma}^2 - \mathbf{H}^2\boldsymbol{\phi}\|_{\mathbf{W}}^2 + \rho\|\boldsymbol{\phi}\|_{\mathbf{I}}^2 \tag{16}$$

where, for every vector $\mathbf{t}$ and positive-definite matrix $\mathbf{A}$, the weighted Euclidean norm is defined as $\|\mathbf{t}\|_{\mathbf{A}} = \sqrt{\mathbf{t}^{\mathrm{T}}\mathbf{A}\mathbf{t}}$. A reasonable choice for the weight matrix $\mathbf{W}$ is a diagonal matrix storing weights proportional to the target variance: $\{\mathbf{W}\}_{i,i} = \{\boldsymbol{\sigma}^2\}_i$. With this choice, we accept a constant relative error. The second term in the right-hand-side of Eq.16 is introduced as a normalization constant for preventing over-fitting (Hastie *et al.*, 2009): $\mathbf{I}$ is the identity matrix and ρ a penalty term. We seek an optimal solution, $\boldsymbol{\phi}^*$, satisfying

$$\begin{aligned} &\boldsymbol{\phi}^* = \operatorname{argmin}_{\boldsymbol{\phi}} C(\boldsymbol{\phi}) \\ &\text{s.t.} \quad \forall i: \phi_i \geq 0 \end{aligned} \tag{17}$$

Eq.17 defines a quadratic cost function and all constrains in Eq.17 are linear. Thus, the problem can be efficiently solved by the available techniques of quadratic programming (Dostal, 2009).

NON COSTANT PEAK FACTOR

In the above formulations we assumed a constant peak factor. An early expression for the peak factor derived by Davenport (1964) is

$$p(\omega) = \sqrt{2\log[\nu(\omega)\tau]} + \frac{0.577}{\sqrt{2\log[\nu(\omega)\tau]}} \tag{18}$$

where τ is the duration of the excitation (taken as the duration of the strong-motion phase of the ground motion, say τ = 10-20s) and $\nu(\omega) = \sqrt{\lambda_2(\omega)/\sigma^2(\omega)}/\pi$ is the mean zero-crossing rate, wherein $\lambda_2(\omega) = \int_0^\infty \gamma^2|H(\omega,\gamma)|^2\phi(\gamma)d\gamma$ is the 2nd spectral moment. To avoid unreasonable values of the peak factor for small frequencies, $p(\omega) = 1.56$ is assumed for $\nu(\omega)\tau < 1.6$. (Otherwise, Eq.19 predicts a peak factor increasing with decreasing frequency.)

Equation 18 can be applied directly, obtaining function $p(\omega)$ for any assigned $\phi(\omega)$. However, if we substitute Eq.18 in Eq.1, we end up with a highly non-linear integral equation. In general, we can solve the problem iteratively: Assume an initial peak factor function $p_{(1)}(\omega)$ and obtain the a corresponding variance function $\sigma^2_{(1)}(\omega) = [A(\omega)]^2/\left[p_{(1)}(\omega)\right]^2$. Any of the methods described in this paper can be applied to this variance function, obtaining a first approximation of the PSD, $\phi_{(1)}(\omega)$. From this, we obtain an improved peak factor function $p_{(2)}(\omega)$ by Eq.18. This process is repeated until convergence is achieved. The approach by Der Kiureghian and Neuenhofer is already based on an iterative scheme, so we can directly combine the updating of the peak factor function in the computation of the corresponding variance, obtaining, instead of Eq.15:

$$\phi_{(k+1)}(\omega) = \frac{[A(\omega)]^2}{\sigma^2_{(k)}(\omega)\left[p_{(k)}(\omega)\right]^2}\phi_{(k)}(\omega) \tag{19}$$

COMPARISON OF METHODS

In this section, we apply the methods presented above to the admissible response spectrum shown in Figure 1. Figure 2 shows the result of the incremental approach by Vanmarke. The target spectrum is plotted in graph (a) (continuous line) together with the reconstructed response spectrum (dash-dotted line). The agreement is not perfect because of the simplifying assumptions behind the method; however, the quality of the approximation should be acceptable for most applications. The peak factor function is reported in graph (b). It grows from 1.9 up to a maximum of 3.55 at a natural frequency of 37Hz; beyond that, the peak factor goes down to the value for the peak ground acceleration, which is 3.36. The influence of the peak factor on the PSD is apparent in graph (c), where the local maximum at 62Hz is related to the decay of the peak factor. Numerically, we face the following issue: If we iterate computations of the peak factor function and of the PSD, the sequence does not converge to a stable solution, but oscillates with period 2. The results in graphs (b) and (c) are obtained after 100 iterations. In summary, we are not able to find a compatible pair of functions $(p(\omega), \phi(\omega))$ so that Eqs.1 and 18 are both accurately satisfied. However, employing the approximate peak factor function depicted in graph (b), the PSD shown in graph (c) is compatible with the target response spectrum.

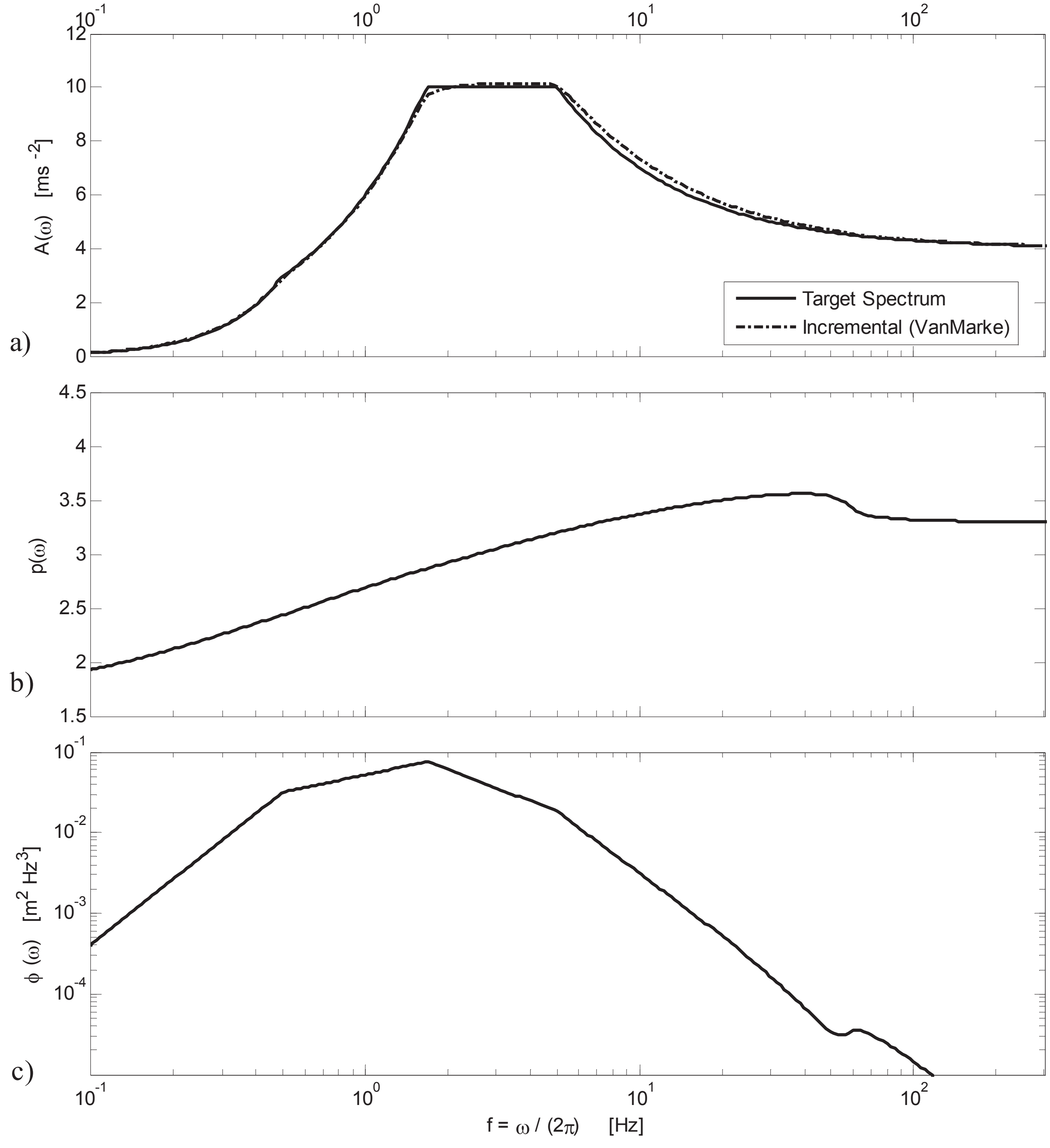

Figure 2. Target (continuous line) and reconstructed (dash-dotted line) response spectra (a), peak factor function (b), identified PSD (c) for the incremental method of Vanmarke.

Figure 3 shows the results of the numerical scheme based on quadratic programming. The agreement between the target and reconstructed spectra shown in graph (a) is much closer, so that the two lines cannot be distinguished. This method is more flexible and does not require the simplifying assumptions of the incremental method – hence the reasons for the close match. The identified PSD, shown in graph (b), exhibits peaks at frequencies where the target response spectrum is not differentiable. These small peaks allow the reconstructed spectrum to closely follow the target one around these points. We experience the same lack of convergence in the peak-factor function as with the incremental method. The optimization was performed by using MATLAB routines. This method is computationally much more demanding than the previous method, and faces the same issue about lack of convergence of the peak factor function.

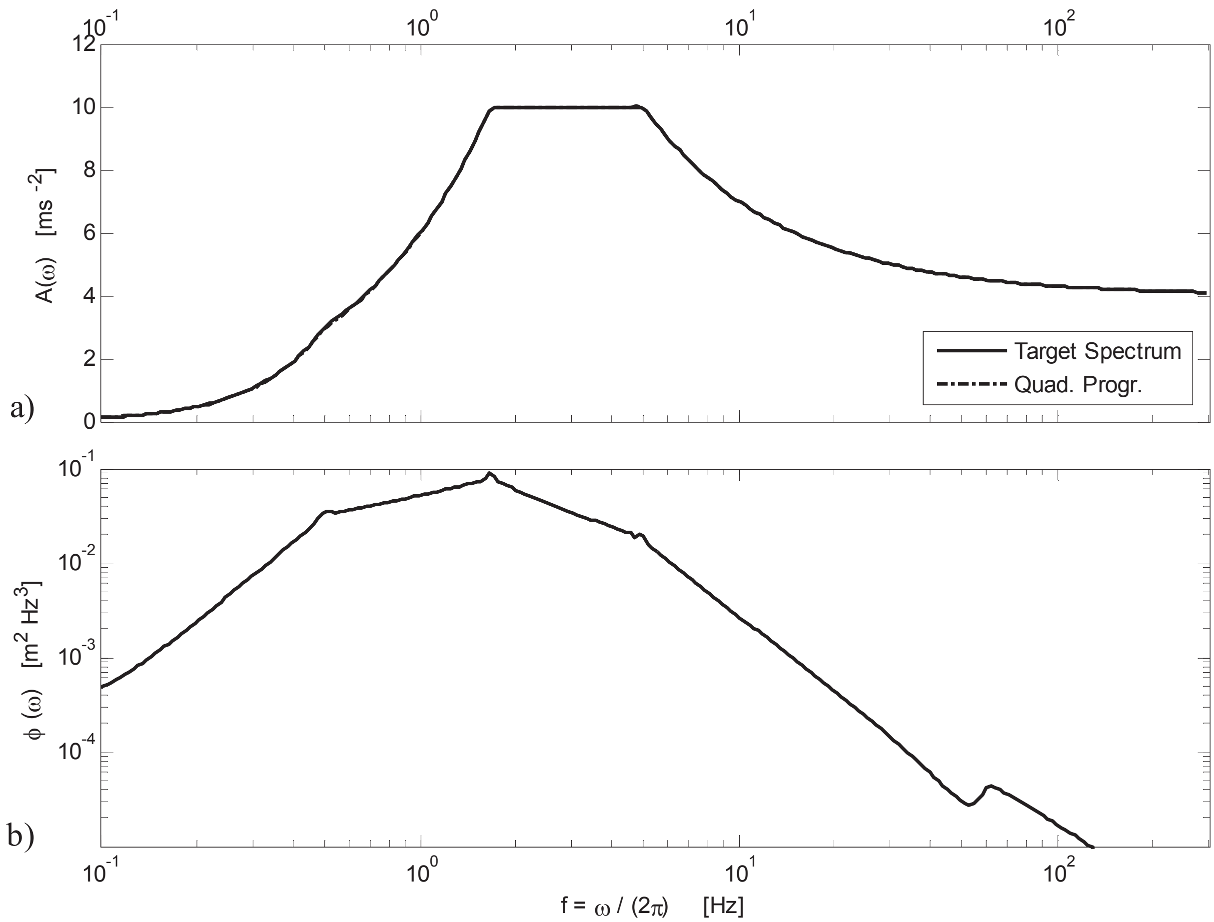

Figure 3. Target (continuous line) and reconstructed (dash-dotted line) response spectra (a), identified PSD (b) for the method based on quadratic programming.

Figure 4 shows the results obtained by the iterative method of Der Kiureghian and Neuenhofer. In graph (a) we show results for iteration numbers 0, 1 and 50. While the agreement is poor at the initial stage, the sequence converges to the target spectrum. In graph (b), we report the correspondent sequence of peak-factor functions and in graph (c) the identified PSDs. Computationally, the method is not as demanding as that based on quadratic programming, but it is more demanding than the incremental method, since it requires solving the forward problem by numerical integration. In this case, we do not experience lack of converge of the sequence of peak-factor functions. The correspondent PSD does not exhibit the local maximum at high frequencies, and the peak-factor function decays more smoothly than those presented in Figures 1(b) and 2(b).

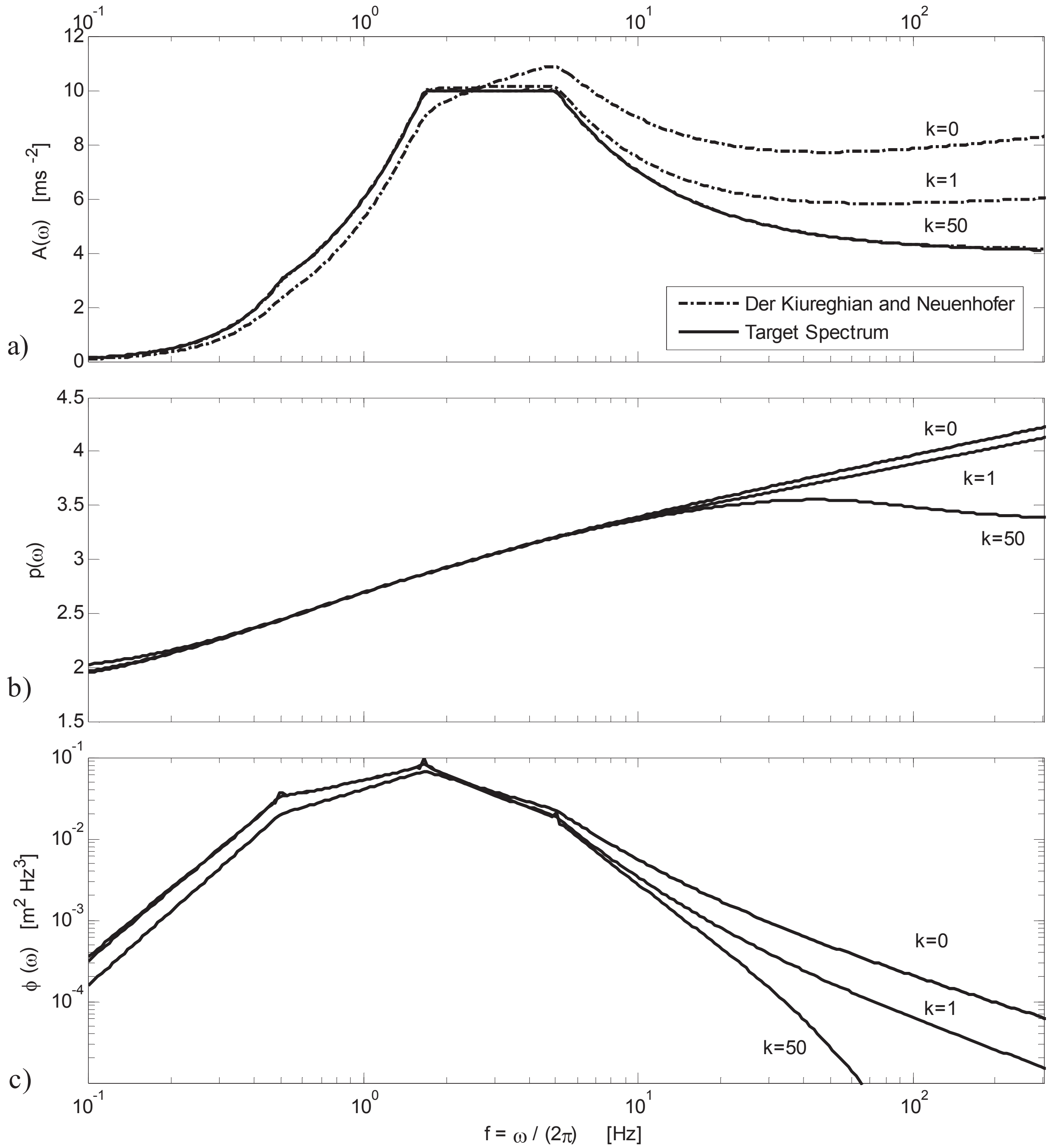

Figure 4. Target (continuous line) and reconstructed (dash-dotted line) response spectra (a), peak factor function (b), identified PSD (c) for the iterative method of Der Kiureghian and Neuenhofer.

SUMMARY AND CONCLUSIONS

Three methods for generating a power spectral density (PSD) compatible with a target response spectrum are described. It is shown that the target response spectrum should satisfy certain requirements to allow generation of spectrum-compatible PSD. A response spectrum not satisfying these conditions violates fundamental rules of random vibrations theory (under the assumption of stationary input and output). This finding provides a basis for examining the admissibility of response spectrum shapes prescribed in building codes.

Of the three methods examined, the incremental procedure by Vanmarke is computationally inexpensive and works well for low damping ratio. The iterative procedure proposed by Der Kiureghian and Neuenhofer works well with admissible target response spectra. Quadratic programming provides a brute-force optimization scheme for solving the inverse problem. It allows for flexibility, selecting an optimal trade-off between smoothness of PSD and accuracy of the reconstructed response spectrum; however, tuning of the weight matrix $\mathbf{W}$ and parameter ρ requires some experience. All three methods produced compatible PSDs for an admissible response spectrum for the entire range of frequencies.

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