Structural Parameter Estimation Using Partial Measurements in Subspace System Identification

Seung-Keun Park, and Hae Young Noh

Abstract— This paper introduces a new structural parameter estimation method for incomplete set of measurements using subspace system identification (subspace SI). Subspace SI initially identifies system matrices that are defined in arbitrary spaces and then transforms them into physical spaces to obtain structural parameters, such as stiffness, damping matrices, and flexural rigidity. This process involves similarity transform which requires measurements from all degrees-of-freedom (DOFs) in the structure to obtain a unique solution. However, in practice it is often difficult, if not impossible, to obtain information from all DOFs due to various constraints, such as sensing systems, noise, budget, etc. This partial measurement case causes the similarity transform problem to become undetermined and have infinite solutions. To address this problem, we developed a method that incorporates both measurement data and prior information about structural parameters within the subspace SI framework to achieve a unique similarity transformation matrix. This matrix is obtained by minimizing the discrepancy between the transformed system equations based on prior information and the measurements. Using this similarity transformation matrix, structural parameters are estimated in physical spaces and updated from the prior information. This updating process can be repeated for various applications, such as model calibration and continuous structural health monitoring. We first investigated the uniqueness of the similarity transformation matrix for full and partial measurement cases and derived solutions. Then, the method is evaluated through a numerical example of a five-story structure, considering several different accuracies of prior information. The results show that our method can well estimate stiffness matrices even with 40% of full DOF measurements. The effects of the locations and the number of measured DOFs on the accuracy of the estimation are investigated as well.

I. INTRODUCTION

Civil infrastructures are constantly exposed to degradation during its lifetime due to normal operational loads (e.g., traffic, wind, temperature variation, etc.) and extreme events (e.g., earthquake, lightening, hurricane, etc.). To insure safety and functionality of these structures, it is critical to accurately and reliably assess structural conditions. Thus, structural integrity needs to be regularly evaluated and updated. Advances in sensor technologies, wireless communications, and computation in the last few decades have enabled us to monitor structural degradation/damage conditions in near real-time.

Structural damage detection methods take either data-driven or physical model-based approaches. In data-driven approaches, damage features are extracted using time-series and signal processing analyses, such as autoregressive model (AR) [1, 2, 3], Fourier transform [4, 5] and wavelet transform [6, 7]. From the extracted damage features, structural damage is diagnosed using statistical pattern recognition methods [1, 2, 3, 8]. Data-driven approaches are often computationally efficient and do not require extensive prior knowledge about structure. On the other hand, physical model-based approaches often require prior information about the structure, need structural responses from multiple locations simultaneously, and are computationally expensive. However, they are based on physical and intuitive parameters and thus provide more detailed structural information, such as locations and extents of damage.

In physical model-based approaches, structures are often modeled using finite element (FE) methods where structural parameters are considered as unknown. These structural parameters are identified by finding the values that best-fit measured responses. Physical model-based approaches are classified according to the form of the governing equation used in system identification (SI), such as second order equation of motion and state-space equation. A second order equation of motion is employed in equation error estimator (EEE) based methods [9, 10] and output error estimator (OEE) based methods [10, 11, 12], while a state-space equation is used in extended Kalman filter [13] and subspace system identification (subspace SI) [14, 15, 16]. EEE and OEE are based on the minimization problem defined by least square errors, and the physical parameters are estimated by solving the minimization problem. EEE has the benefit of not requiring initial conditions. In addition, if all state vectors (displacement, velocity, and acceleration) from all degrees-of-freedom (DOFs) are measured, the minimization problem becomes linear with respect to structural parameters. However, EEE is known to be a biased estimator, which means the bias of the estimated parameters does not decrease even though more data become available [10]. The benefits of OEE are as follows: it does not require the measurements from all DOFs and it is an unbiased estimator. Yet, it requires initial conditions and sensitivity analysis of dynamic responses with respect to structural parameters [11, 12], because the minimization problem is nonlinear with respect to the structural parameters. On the other hand, subspace SI that uses a state-space equation as a governing equation provides an unbiased estimator and does not require initial conditions and sensitivity information with
respect to structural parameters. It also does not require having all three types of measurements, such as displacement, velocity, and acceleration to keep the system linear. These benefits make subspace SI more applicable in practice.

Subspace SI has been developed for identification of linear time-invariant state-space models [14, 15, 16]. Subspace SI first estimates system matrices that describe a state-space model and then extracts modal parameters, such as natural frequencies, modal damping ratios, and mode shapes, sampled at measurement locations [18, 19]. Although the estimated system matrices and modal parameters are useful for the control algorithm development, they cannot be directly applied to structural system identification for detailed damage localization and quantification. This is because the estimated system matrices are located in an arbitrary space and do not have physical meanings. To extract the structural parameters in physical spaces, such as stiffness and flexural rigidities, the system matrices and the state vector of the state-space model should be transformed into the physical space using proper similarity transformation. Similarity transformation is a coordinate transformation for system matrices and a state vector that does not change the transfer function or matrix of the structural system. Using the transformed system matrices, structural parameters, such as stiffness and damping matrices, are estimated.

Various methods to calculate the similarity transformation matrix from state-space model based system identification have been developed [20, 21, 22] and applied to structural system identification [23, 24]. In general, the unique solution for the similarity transformation matrix is obtained by finding the one that transforms the estimated system matrices into physical canonical forms [20, 21]. However, the aforementioned approaches require measurements from all DOFs in the structure. Hereafter, the measurements from a complete and an incomplete set of DOFs are referred to as full and partial measurements, respectively. In practice, it is difficult, if not impossible, to measure structural responses from all DOFs due to the costs of data acquisition system, sensing constraints, missing data problem, and/or excessive number of DOFs like civil infrastructures. This significantly limits the applicability of conventional subspace SI methods in structural health monitoring. In particular, if a structure is densely instrumented, it is desirable to decentralize data processing to reduce communication delays and computational burden.

Given this challenge, the main objective of this paper is to develop a method to obtain structural parameters with partial measurements using subspace SI by incorporating measurement data and prior information. In partial measurement cases, the problem to obtain the similarity transformation matrix turns into an underdetermined problem due to the lack of constraints from missing measurements. Thus, the similarity transformation matrix cannot be determined uniquely [20, 22]. To address this problem, we combine the system matrices estimated from measurement data and the ones obtained from prior information (model) about the structure. The similarity transformation matrix is obtained by minimizing the discrepancy between the transformed system equation based on the prior model and the one based on the measurements. This can be interpreted as structural parameter updating from the prior model using the measurements. This minimization problem is constrained by a set of linear equations, and the similarity transformation is uniquely determined by solving these equations. Based on this similarity transformation matrix, system matrices are transformed into physical spaces and structural parameters are extracted. This updating process can be repeated for various applications, such as model calibration and continuous structural health monitoring.

The remainder of this paper is organized as follows. Section II describes the relationship between second order dynamic equation of motion and state-space model for extracting structural parameters using conventional subspace SI. Section III investigates the uniqueness of the similarity transformation matrix and derives the method to obtain the matrix using only partial displacement or acceleration measurements. In section IV, our method is applied to a numerical simulation study of a five-story shear building with varying numbers and locations of measured DOFs for different accuracies of prior model. Finally, section V provides summary and conclusions of the paper.

II. THEORY FOR IDENTIFYING STRUCTURAL PARAMETERS

The relationship between the second order dynamic equation of motion and a state-space model is introduced in this section. An equation of motion at time $t$ for linear time-invariant structural system is expressed as

$$ M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = B_\nu u(t) $$

where $M$, $C$, $K$, and $u$ are $n \times n$ mass, damping, stiffness matrices of a structure, and a $m$-dimensional external force vector, respectively. $\dot{q}$, $q$, and $\ddot{q}$ denote $n$-dimensional acceleration, velocity, and displacement vector, respectively. The $n \times m$ matrix $B_\nu$ is the input influence matrix, whose components are unity at forced DOFs and zero at unforced DOFs.

Equation (1) can be expressed by continuous-time state-space form as

$$ \dot{x}(t) = A_x x(t) + B_x u(t), \quad y(t) = C_x x(t) + D_x u(t) $$

(2)
where $A_a$, $B_a$, $C_a$ and $D_a$ are system matrices, while $x$ and $y$ denote a $2n$-dimensional state vector and a $p$-dimensional output vector (measurements), respectively. The dimensions of the system matrices $A_a$, $B_a$, $C_a$ and $D_a$ are $2n \times 2n$, $2n \times m$, $p \times 2n$, and $p \times m$, respectively. $A_b$, $B_b$, and $x$ can be represented as

$$
A_b = \begin{bmatrix}
0 & I_s \\
-M^aK & -M^aC_s
\end{bmatrix},
B_b = \begin{bmatrix}
0
\end{bmatrix},
x = \begin{bmatrix}
q
\end{bmatrix}.
$$

Depending on the type of sensors, $C_c$ and $D_c$ have appropriate forms. For displacement measurements,

$$
C_c = \begin{bmatrix}
C_d
0_{pxm}
\end{bmatrix},
D_c = 0_{pxm}
$$

where $C_d$ represents a $p \times n$ location matrix for displacement sensors. For acceleration measurements,

$$
C_c = C_d[-M^{-a}K - M^{-a}C],
D_c = C_M^{-1}B_c
$$

where $C_M$ is a $p \times n$ location matrix for accelerometers.

Subspace SI extracts system matrices from the state-space model and then converts them to structure parameters using similarity transformation, as shown in Fig. 1. Fig. 1(a) shows that subspace SI estimates the system matrix $\hat{A}_b$, $\hat{B}_b$, $\hat{C}_b$, $\hat{D}_b$ and state vector $\hat{x}$ from input vector $u$ and output vector $y$ using matrix decomposition methods. However, the estimated system matrices and state vector that satisfy (2) for given $u$ and $y$ are defined in an arbitrary space and are not unique. In other words, the system matrices and state vector set $(T\hat{A}_b, T^{-1}, T\hat{B}_b, T\hat{C}_b, T^{-1}, T\hat{D}_b, T\hat{x})$ with any similarity transformation matrix $T$, which is an invertible matrix, can be a solution of (2). In order to extract physical structural parameters from the identified system matrices, we need to find the unique transformation matrix $T$ that can convert the system matrices in the arbitrary space into those in the physical space. The transformed system matrices should satisfy the constraints shown in the gray boxes in Fig. 1. When full measurements are available, the similarity transformation can be uniquely obtained by using only these constraints [20, 21, 23] for displacement, velocity, and acceleration measurements as shown in Fig. 1(b). After transforming into the physical space, $A_a$ and $C_a$ can be shown as partitioned matrices as

$$
A_a = \begin{bmatrix}
0_{n \times m} & I_s \\
E_1 & E_2
\end{bmatrix},
C_a = \begin{bmatrix}
C_d
0_{pxm}
\end{bmatrix}
$$

for displacement measurements

$$
A_a = \begin{bmatrix}
0_{n \times m} & I_s \\
E_1 & E_2
\end{bmatrix},
C_a = C_d[-M^{-a}K - M^{-a}C]
$$

for acceleration measurements

(6.a)

(6.b)

where $E_1$ and $E_2$ are $n \times n$ matrices. From $E_1$ and $E_2$, stiffness and damping matrices can be easily calculated as

$$
K = -M^{-1}E_1,
C = -ME_2
$$

assuming that the mass matrices are known.

If only partial measurements are available, the similarity transformation matrix cannot be uniquely determined by the constraints in gray boxes. Therefore, additional constrains are imposed by prior information in this paper. The similarity
transformation matrix is calculated by finding \( T \) to minimize the error between transformed system matrix \( \hat{A}_n \) and the prior system matrix \( A_p \) under the constraints in gray boxes in Fig. 1. The detailed procedure is explained in section III.

### III. PHYSICAL PARAMETERS EXTRACTION WITH PARTIAL MEASUREMENTS

This section describes our method to extract structural parameters using partial measurements. We first investigate the uniqueness of the similarity transformation in full and partial measurement cases. Then, the method to derive a similarity transformation matrix using partial measurements is introduced for displacement and acceleration measurements, respectively.

#### A. Uniqueness of the Similarity Transformation

As a simplest example, consider the displacement as measurements. Fig. 2 summarizes the method to compute the similarity transformation matrix. According to (6.a), the similarity transformation matrix \( T \) needs to satisfy

\[
T\hat{A}_n T^{-1} = \hat{A}_p, \quad \hat{C}_n T^{-1} = \hat{C}_p = [C_d \quad 0_{p \times n \times \ldots}]
\]

where \( \hat{A}_p \) and \( \hat{C}_p \) are the system matrices estimated by subspace system identification. \( T, \hat{A}_p, \) and \( \hat{C}_p \) are further represented as

\[
T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad \hat{A}_p = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{C}_p = [\hat{C}_1 \quad \hat{C}_2, \ldots] \tag{9}
\]

Substituting (9) into (8) results in

\[
\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \begin{bmatrix} 0_{n \times p} & I_{n \times n} \\ E_1 & E_2 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}
\]

\[
[C_{d_p}]_{0 \times n \times p} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = [\hat{C}_1 \quad \hat{C}_2, \ldots] \tag{10.b}
\]

The sub-matrix \([0 \quad I]\) in \( A_p \), which represents the relationship between the time derivative of state vector \( \dot{x} \) and state vector \( x \) in the state space model, and the matrix \( C_p = [C_d \quad 0] \), which denotes the relationship between the state vector \( x \) and output vector \( y \) do not depend on structural parameters. Therefore, they (also shown in gray boxes in Fig. 2) need to be strictly satisfied in any cases. Given this, we can derive an equality constraint using the equations in the box (a) at the bottom of Fig. 2 as

\[
T_{11} \hat{A}_{11} + T_{12} \hat{A}_{12} = T_{11}, \quad T_{11} \hat{A}_{11} + T_{12} \hat{A}_{12} = T_{21}, \quad C_p T_{11} = \hat{C}_1, \quad C_p T_{12} = \hat{C}_2
\]

Using the Kronecker product \( \otimes \) and a property of the Kronecker product \( \text{vec}(ABD) = (B^T \otimes A) \vec{D} \), the unknown matrix \( T \) is converted to a vector form [25]. The definition of the Kronecker product \( \otimes \) is given as

\[
A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{12}
\]
where \(a_{ij}\) is the element of the matrix \(A\) in the \(i^{th}\) row and \(j^{th}\) column. Note that if \(A\) and \(B\) are \(m\times n\) and \(p\times q\) matrices, respectively, the dimension of \(A \otimes B\) is \(mp\times nq\). \(\text{vec}D\) is a vectorized matrix \(D\) given as

\[
\text{vec}D = \begin{pmatrix}
d_{11} & d_{21} & \cdots & d_{m1} & d_{12} & d_{22} & \cdots & d_{m2} & \cdots & d_{1n} & d_{2n} & \cdots & d_{mn}
\end{pmatrix}^T
\]  

where \(d_{ij}\) represents the element of the matrix \(D\) at \(i^{th}\) row and \(j^{th}\) column. Equation (11) is expressed using \(\text{vec}T\) as

\[
(\hat{\mathbf{A}}_{11} \otimes \mathbf{I}_n) \text{vec}T_{11} + (\hat{\mathbf{A}}_{21} \otimes \mathbf{I}_n) \text{vec}T_{12} = \text{vec}T_{11} = 0
\]  

(14.a)

\[
(\hat{\mathbf{A}}_{12} \otimes \mathbf{I}_n) \text{vec}T_{11} + (\hat{\mathbf{A}}_{22} \otimes \mathbf{I}_n) \text{vec}T_{12} = \text{vec}T_{12} = 0
\]  

(14.b)

\[
(\mathbf{I}_n \otimes \mathbf{C}_p) \text{vec}T_{11} = \text{vec}\hat{\mathbf{C}}_1
\]  

(14.c)

\[
(\mathbf{I}_n \otimes \mathbf{C}_p) \text{vec}T_{12} = \text{vec}\hat{\mathbf{C}}_2
\]  

(14.d)

The four relationships in (14) are necessary conditions to satisfy for \(T\). The total number of unknowns is identical to the number of elements in \(T\), which is \(4n^2\). The number of equations at (14.a), (14.b), (14.c) and (14.d) are \(n^2\), \(n^2\), \(pn\) and \(pn\), respectively. Therefore, the total number of equations is \(2n^2+2pn\). For full measurement cases \((p=n)\), since the number of unknowns is same as the number of equations, the transformation matrix \(T\) is determined uniquely by using the equality constraints

\[
\begin{bmatrix}
G_{11} & 0 & 0 & 0 \\
0 & G_{11} & 0 & 0 \\
G_{21} & G_{22} & -\mathbf{I} & 0 \\
G_{22} & 0 & -\mathbf{I} & \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\text{vec}T_{11} \\
\text{vec}T_{12} \\
\text{vec}T_{11} \\
\text{vec}T_{12}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\mathbf{I} \\
-\mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\text{vec}\hat{\mathbf{C}}_1 \\
\text{vec}\hat{\mathbf{C}}_2 \\
0 \\
0
\end{bmatrix}
\]  

(15)

\(G, \tau = d\)

where \(G_{11} = \mathbf{I}_n \otimes \mathbf{C}_p\), \(G_{21} = \hat{\mathbf{A}}_{11} \otimes \mathbf{I}_n\), \(G_{12} = \hat{\mathbf{A}}_{12} \otimes \mathbf{I}_n\), \(G_{22} = \hat{\mathbf{A}}_{22} \otimes \mathbf{I}_n\), and \(G_{22} = \hat{\mathbf{A}}_{22} \otimes \mathbf{I}_n\). For partial measurement cases \((p<n)\), since the number of equations is smaller than that of unknowns, \(T\) is not uniquely determined. In this case, additional \(2n^2-2pn\) constraints are required to obtain \(T\). Thus, the equations in the box (b) at the bottom of Fig. 2 are used as the additional condition \((\mathbf{G}, \tau = \mathbf{e})\) for solving this rank-deficient problem.

\section*{B. Similarity Transformation for Partial Measurements: Displacement}

In conventional methods for calculating \(T\) \([20, 21, 23]\), \(E_1\) and \(E_2\) in (8) are unknown matrices and are not used to determine \(T\). For partial measurement cases, we used stiffness and damping matrices from prior model, denoted as \(K_p\) and \(C_p\), respectively, to construct \(E_1\) and \(E_2\), which are used as additional constraints. Substituting \(E_1\Rightarrow -\mathbf{M}^T\mathbf{K}_p\) and \(E_2\Rightarrow -\mathbf{M}^T\mathbf{C}_p\) into (8), we obtain

\[
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
\hat{\mathbf{A}}_{11} & \hat{\mathbf{A}}_{12} \\
\hat{\mathbf{A}}_{21} & \hat{\mathbf{A}}_{22}
\end{bmatrix} =
\begin{bmatrix}
\mathbf{0} \\
\mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\mathbf{T}^T_{11} & \mathbf{T}^T_{12} \\
\mathbf{T}^T_{21} & \mathbf{T}^T_{22}
\end{bmatrix}
\]  

(16)

From the bottom row of (16),

\[
\mathbf{M}^\mathbf{T}\mathbf{K}_p\mathbf{T}_{11} + \mathbf{M}^\mathbf{T}\mathbf{C}_p\mathbf{T}_{21} + \mathbf{T}_{11}\hat{\mathbf{A}}_{11} + \mathbf{T}_{21}\hat{\mathbf{A}}_{21} = 0_{n\times n}
\]  

(17.a)

\[
\mathbf{M}^\mathbf{T}\mathbf{K}_p\mathbf{T}_{12} + \mathbf{M}^\mathbf{T}\mathbf{C}_p\mathbf{T}_{22} + \mathbf{T}_{12}\hat{\mathbf{A}}_{12} + \mathbf{T}_{22}\hat{\mathbf{A}}_{22} = 0_{n\times n}
\]  

(17.b)

If there is no modeling error for the prior model and no measurement error, (17) is exactly satisfied. However, due to modeling and measurement errors, it is very difficult in practice to exactly satisfy (17), which leads to residual errors of

\[
\mathbf{M}^\mathbf{T}\mathbf{K}_p\mathbf{T}_{11} + \mathbf{M}^\mathbf{T}\mathbf{C}_p\mathbf{T}_{21} + \mathbf{T}_{11}\hat{\mathbf{A}}_{11} + \mathbf{T}_{21}\hat{\mathbf{A}}_{21} = \mathbf{e}_1
\]  

(18.a)

\[
\mathbf{M}^\mathbf{T}\mathbf{K}_p\mathbf{T}_{12} + \mathbf{M}^\mathbf{T}\mathbf{C}_p\mathbf{T}_{22} + \mathbf{T}_{12}\hat{\mathbf{A}}_{12} + \mathbf{T}_{22}\hat{\mathbf{A}}_{22} = \mathbf{e}_2
\]  

(18.b)

where \(\mathbf{e}_1\) and \(\mathbf{e}_2\) are residual matrices. Using the Kronecker product, (18) is converted to vector form as

\[
(\mathbf{I} \otimes \mathbf{M}^\mathbf{T}\mathbf{K}_p) \text{vec}T_{11} + [(\mathbf{I} \otimes \mathbf{M}^\mathbf{T}\mathbf{C}_p) + (\hat{\mathbf{A}}_{11} \otimes \mathbf{I})] \text{vec}T_{21} + (\hat{\mathbf{A}}_{21} \otimes \mathbf{I}) \text{vec}T_{22} = \text{vec}\mathbf{e}_1
\]  

(19.a)

\[
(\mathbf{I} \otimes \mathbf{M}^\mathbf{T}\mathbf{K}_p) \text{vec}T_{12} + (\hat{\mathbf{A}}_{12} \otimes \mathbf{I}) \text{vec}T_{21} + [(\mathbf{I} \otimes \mathbf{M}^\mathbf{T}\mathbf{C}_p) + (\hat{\mathbf{A}}_{22} \otimes \mathbf{I})] \text{vec}T_{22} = \text{vec}\mathbf{e}_2
\]  

(19.b)

Equation (19) is converted to the matrix form
where \( G_{sa} = I_n \otimes M^+K_p \), \( G_{sv} = I_n \otimes M^+C_p \). To obtain the similarity transformation matrix, we search for \( T \) that minimizes the norm of the residual matrices in (20) while imposing the strict equality constraints in (15) as

\[
\min \, \Pi = \frac{1}{2} (\text{vec} e)^T \text{vec} e = \frac{1}{2} \tau^T G\hat{c} \tau \quad \text{subject to} \quad G \tau = d
\]

(21)

In other words, the estimated \( T \) results in the stiffness and damping matrices those are closest to the prior information while satisfying the constraints (15). The solution to (21) is obtained by the following linear algebraic equation [27]

\[
\begin{bmatrix}
G\hat{c}^T G\hat{c} & G\hat{c}^T \\
G\hat{c} & 0
\end{bmatrix}
\begin{bmatrix}
\tau \\
\lambda
\end{bmatrix} = 0
\]

(22)

where \( \lambda \) is a Lagrange multiplier. Since \( \tau \) is a vector form of \( T \), \( T \) is reconstructed from estimated \( \tau \). Through the transformation, \( A_\tau \) in the physical space is computed and the stiffness and damping matrices are estimated using (7).

C. Similarity Transformation for Partial Measurements: Acceleration

In many structural health monitoring applications, acceleration measurements are often used as outputs of a state space model. In this case, the transformation matrix needs to satisfy

\[
A_\tau = TA_\tau T^{-1}
\]

(23a)

\[
C_\tau = C_\tau [-M^{-1}K - M^{-1}C] = CT^{-1}.
\]

(23b)

Equation (23a) results in the same equations as the displacement measurements case, which are (14.a), (14.b) and (20). The difference from the displacement measurement case come from (23.b) as

\[
C_\tau [-M^{-1}K - M^{-1}C] \begin{bmatrix}
T_{11} \\
T_{12} \\
T_{21} \\
T_{22}
\end{bmatrix} = \hat{C} = [\hat{C}_1 \hat{C}_2]
\]

(24)

Since \( K \) and \( C \) are unknown, we cannot directly use (24) to obtain \( T \). Instead, we utilize (17). As explained above, (17) is exactly satisfied when \( K_p \) and \( C_p \) are true values. Therefore, premultiplying \( C_\tau \) in (17) yields the following relationships

\[
C_\tau (M^{-1}KT_{11} + M^{-1}CT_{21} + T_{11}\hat{A}_{11} + T_{21}\hat{A}_{21}) = 0
\]

(25a)

\[
C_\tau (M^{-1}KT_{21} + M^{-1}CT_{22} + T_{12}\hat{A}_{12} + T_{22}\hat{A}_{22}) = 0
\]

(25b)

Equation (24) can be further simplified using (25) as

\[
C_\tau (T_{21}\hat{A}_{11} + T_{22}\hat{A}_{12}) = \hat{C}_1
\]

(26a)

\[
C_\tau (T_{21}\hat{A}_{12} + T_{22}\hat{A}_{22}) = \hat{C}_2
\]

(26b)

By using Kronecker product, (26) is converted to

\[
(\hat{A}_{11} \otimes C_\tau) \text{vec} T_{11} + (\hat{A}_{12} \otimes C_\tau) \text{vec} T_{21} = \text{vec} \hat{C}_1
\]

(27a)

\[
(\hat{A}_{12} \otimes C_\tau) \text{vec} T_{21} + (\hat{A}_{22} \otimes C_\tau) \text{vec} T_{22} = \text{vec} \hat{C}_2
\]

(27b)

Then by combining (14.a), (14.b) and (27), the equality constraint equations are obtained as

\[
\begin{bmatrix}
0 & 0 & G_{11} & G_{21} \\
0 & 0 & G_{12} & G_{22} \\
G_{11} & G_{21} & -1 & 0 \\
G_{12} & G_{22} & 0 & -1
\end{bmatrix}
\begin{bmatrix}
\text{vec} T_{11} \\
\text{vec} T_{12} \\
\text{vec} T_{21} \\
\text{vec} T_{22}
\end{bmatrix}
= \begin{bmatrix}
\text{vec} \hat{C}_1 \\
\text{vec} \hat{C}_2 \\
0 \\
0
\end{bmatrix}
\Rightarrow G_{sa} \tau = d
\]

(28)
where \( G'_{11} = \hat{A}'_{11} \otimes C_u \), \( G'_{12} = \hat{A}'_{12} \otimes C_u \), \( G'_{21} = \hat{A}'_{21} \otimes C_u \), and \( G'_{22} = \hat{A}'_{22} \otimes C_u \). Using (20) and (28), the similarity transformation matrix is obtained through the minimization problem, defined in the same manner as (21). This minimization problem can be solved by

\[
\begin{bmatrix}
G'_{22} & G'_{21} \\
G'_{12} & G'_{11}
\end{bmatrix}
\begin{bmatrix}
\tau \\
\hat{\lambda}
\end{bmatrix}
= \begin{bmatrix}
0 \\
\mathbf{d}_s
\end{bmatrix}.
\]  

(29)

Stiffness and damping matrices are estimated in the same way as the displacement measurements case as shown in (7).

IV. NUMERICAL EXAMPLES

The performance of the proposed method is evaluated through numerical simulation studies of a five story shear building. Structural parameters are estimated with various numbers and locations of sensors (i.e., measurements) and degrees of modeling errors in prior information. The five-story shear building used in this numerical simulation study is shown in Fig. 3. Flexural rigidity, mass property of each story, and modal damping ratios are given in Fig. 3. The natural frequencies associated with the first five modes are 0.67, 1.82, 2.85, 3.56, and 4.13 Hz, respectively. Modal damping model is also employed to model the structural dynamics. Generally, the damping ratios increase with a mode number, and they are assumed to be 1, 2, 3, 4 and 5% for the 1st through 5th modes, respectively. Horizontal random excitation at each story of the building is used as an input to the system, which is a uniform distribution with its mean, maximum, and minimum values as 0, 10, and -10 N, respectively. The corresponding acceleration responses are used as outputs. The 5% random proportional noise is added to the measurement responses to simulate measurement noise, whose mean is zero and the maximum magnitude is 5% of the magnitude of the response. The sampling rate is 100 Hz, and the length of the data is 5.0 seconds. In order to simulate changes in structural parameters (i.e., inaccuracy in prior model), structural damage scenarios are introduced by reducing 30% and 50% of the flexural rigidity at the first and third story. The prior model provides the stiffness and damping matrices at the initial intact state. To evaluate the accuracy of the estimation, the flexural rigidity of each story is calculated from the estimated stiffness matrix and compared with the true values [26]. Note that this method can be also applied to estimate damping and stiffness matrices, but due to space limitation, we present the results for only flexural rigidities in this paper.

In order to quantify the estimation accuracy, estimation errors are defined as

\[
\mathcal{E} = \frac{\|\mathbf{k}_{\text{est}} - \mathbf{k}_{\text{exact}}\|}{\|\mathbf{k}_{\text{exact}}\|},
\]  

(30)

where \( \mathbf{k}_{\text{est}} \) and \( \mathbf{k}_{\text{exact}} \) denote the estimated flexural rigidity and the exact flexural rigidity. Thus, \( \mathcal{E} \) represents how accurate the estimation is compared to the true value.

1) 80% partial measurement case (Four DOFs)

This section considers the case when only 80% of measurements are available (i.e., four out of five sensor data). Two damage scenarios (DS1, DS2) are simulated by 30% stiffness reduction at the first and third stories, and corresponding flexural rigidities are given in Table I.
Figure 4. Estimated flexural rigidities for various sensor location cases for 30% damage with 80% partial measurements

There are five different combinations of sensor locations as shown in Table II. Figure 4 shows the flexural rigidity identification results for five different sensor location cases. The vertical axis of Fig. 4 represents the normalized flexural rigidity with respect to that of the intact state. The estimation errors defined in (30) are calculated for each case as shown in Table II. When the damage location is on the first story (DS₁), Cases I, II, and III yield 1.13%, 1.02% and 0.82% for the estimation error (\(\bar{\epsilon}\)), respectively. In these cases, the identified flexural rigidity of the first story member is close to the exact value (i.e., 0.7) and those of the other stories are close to 1.0 as shown in Fig. 4(a). In Cases IV and V, \(\bar{\epsilon}\)’s are increased to 4.37% and 5.08%, respectively. As the sensor locations are closer to the damage location, the estimation becomes more accurate. When the third story is damaged (DS₃), Cases I and II yield very accurate results, where \(\bar{\epsilon}\)’s are 1.05% and 3.47%. For Cases III, IV, and V, \(\bar{\epsilon}\)’s are increased to 6.90%, 7.86% and 8.77%. Even though we miss measurements from one location and the prior information is different from the true structural state with damage, the average \(\bar{\epsilon}\) values are very low for both damage cases, 2.48% and 5.61%, respectively.

To further investigate the influence of sensor location, sensitivity analysis is performed. Table III shows the sensitivities of \(i^{th}\) mode shape (\(\phi_i\)) with respect to \(j^{th}\) flexural rigidity (\(k_j\)) for \(i = 1, 2, \ldots, 5\), and \(j = 1\) and 3. The sensitivities are calculated using a finite difference method, given as

\[
\frac{\hat{\phi}_i}{\tilde{k}_j} = \frac{\phi_i(k_j + \Delta k_j) - \phi_i(k_j)}{\Delta k_j}
\]

where \(\Delta k_j = k_j/10^6\). To consider the contribution of each mode, modal coordinates (\(q_j\)) and time average of them are calculated as

\[
a(t) = \sum_{i=1}^{5} \phi_i q_i(t), \quad a_j(t) = \bar{\phi}_j^T Ma(t), \quad \bar{q}_j = \frac{1}{nT} \sum_{t=1}^{nT} (q_j(t))^2
\]

<table>
<thead>
<tr>
<th>Case</th>
<th>Intact State</th>
<th>Damage at First Story (DS₁)</th>
<th>Damage at Third Story (DS₃)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexural Rigidity (MN/m)</td>
<td>(k_{pse} = \begin{bmatrix} 5.0 \ 4.0 \ 3.0 \end{bmatrix} )</td>
<td>(k_{pse} = \begin{bmatrix} 3.5 \ 4.0 \ 3.0 \end{bmatrix} )</td>
<td>(k_{pse} = \begin{bmatrix} 5.0 \ 4.0 \ 3.0 \end{bmatrix} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Case</th>
<th>Measured DOF</th>
<th>Damage Scenario</th>
<th>(\bar{\epsilon})</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>Case II</td>
<td>Case III</td>
<td>Case IV</td>
<td>Case V</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Measured DOF</td>
<td>(1^\circ, 2^\circ, 3^\circ, 4^\circ )</td>
<td>(1^\circ, 2^\circ, 3^\circ, 5^\circ )</td>
<td>(1^\circ, 2^\circ, 5^\circ, 4^\circ )</td>
<td>(2^\circ, 3^\circ, 4^\circ, 5^\circ )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Damage Scenario</td>
<td>DS₁</td>
<td>DS₂</td>
<td>DS₁</td>
<td>DS₂</td>
<td>DS₁</td>
<td>DS₂</td>
<td>DS₁</td>
<td>DS₂</td>
</tr>
<tr>
<td>(\bar{\epsilon})</td>
<td>0.0113</td>
<td>0.0105</td>
<td>0.0102</td>
<td>0.0347</td>
<td>0.0082</td>
<td>0.0690</td>
<td>0.0437</td>
<td>0.0786</td>
</tr>
</tbody>
</table>
The calculated $\ddot{q}_i$ is $[\ddot{q}_1, \ddot{q}_2, \ddot{q}_3, \ddot{q}_4] = [1.65 \times 10^{-3}, 3.21 \times 10^{-3}, 5.58 \times 10^{-3}, 3.93 \times 10^{-3}]$ and represents how much each mode contributes to the acceleration. The weighted averages of the sensitivity at each DOF with respect to flexural rigidity of each member are obtained by using the values in Table III and $\ddot{q}_i$, as

$$\frac{\partial \Phi}{\partial k_i} = \sum_{i=1}^{nq} \frac{\partial \Phi}{\partial q_i} \frac{\ddot{q}_i}{\ddot{k}_i} = \begin{bmatrix} 0.71 \\ 1.02 \\ 0.77 \\ 0.58 \\ 0.21 \end{bmatrix} \times 10^{-12}, \quad \frac{\partial \Phi}{\partial k_i} = \sum_{i=1}^{nq} \frac{\partial \Phi}{\partial q_i} \frac{\ddot{q}_i}{\ddot{k}_i} = \begin{bmatrix} 0.98 \\ 1.01 \\ 0.95 \\ 0.84 \\ 0.67 \end{bmatrix} \times 10^{-12}. \quad (33)$$

For DS$_1$, the sensitivities of the first three DOFs are larger than those of the other DOFs. Therefore, the sensor location Cases I, II, and III yield more accurate results than the other cases. For DS$_2$, since the first three DOFs have larger sensitivities than the others, the Case I and II yield more accurate results than the other cases even though the damage occurs on the third story member.

To assess the model updating performance of our method, the extent of damage for DS$_1$ (30% on the first story) is increased to 50% (i.e., $k_{ds} = [2.5 \ 4.0 \ 4.0 \ 3.0 \ 3.0]^T$), and the estimated flexural rigidities for DS$_1$ are used as a prior model. This damage state is referred to as DS$_3$. Fig. 5 shows the results for five sensor location cases. The estimation error, as defined in (30), is calculated for each case as shown in Table IV.

### Table III. Sensitivities of Mode Shapes with Respect to Flexural Rigidity of Each Member

<table>
<thead>
<tr>
<th>Member Number</th>
<th>$\frac{\partial \Phi}{\partial k_i}$</th>
<th>$\Phi_1$</th>
<th>$\Phi_2$</th>
<th>$\Phi_3$</th>
<th>$\Phi_4$</th>
<th>$\Phi_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>$\frac{\partial \Phi}{\partial k_1} = \begin{bmatrix} -1.65 \ -0.90 \ -0.20 \ 0.51 \ 0.86 \end{bmatrix} \times 10^{-16}$</td>
<td>3.17</td>
<td>-0.68</td>
<td>2.54</td>
<td>-2.63</td>
<td>-1.32</td>
</tr>
<tr>
<td></td>
<td>$\frac{\partial \Phi}{\partial k_2} = \begin{bmatrix} -3.06 \ -1.65 \ 0.32 \end{bmatrix} \times 10^{-16}$</td>
<td>0.33</td>
<td>2.09</td>
<td>2.09</td>
<td>1.29</td>
<td>0.67</td>
</tr>
<tr>
<td>$k_2$</td>
<td>$\frac{\partial \Phi}{\partial k_1} = \begin{bmatrix} 0.65 \ 1.32 \ -0.75 \ -0.31 \ -0.08 \end{bmatrix} \times 10^{-16}$</td>
<td>0.99</td>
<td>-2.05</td>
<td>-2.05</td>
<td>4.02</td>
<td>2.87</td>
</tr>
<tr>
<td></td>
<td>$\frac{\partial \Phi}{\partial k_2} = \begin{bmatrix} -1.74 \ 0.13 \ 1.47 \end{bmatrix} \times 10^{-16}$</td>
<td>1.34</td>
<td>-4.83</td>
<td>1.65</td>
<td>-3.45</td>
<td>3.92</td>
</tr>
</tbody>
</table>

### Table IV. Estimation Errors for 50% Damage with 80% Partial Measurements

<table>
<thead>
<tr>
<th>Case</th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
<th>Case IV</th>
<th>Case V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured DOF</td>
<td>$1^{st}$, $2^{nd}$, $3^{rd}$, $4^{th}$</td>
<td>$1^{st}$, $2^{nd}$, $3^{rd}$, $4^{th}$</td>
<td>$1^{st}$, $2^{nd}$, $3^{rd}$, $5^{th}$</td>
<td>$1^{st}$, $2^{nd}$, $4^{th}$, $5^{th}$</td>
<td>$2^{nd}$, $3^{rd}$, $4^{th}$, $5^{th}$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0135</td>
<td>0.0110</td>
<td>0.0064</td>
<td>0.0461</td>
<td>0.0642</td>
</tr>
</tbody>
</table>
By comparing Fig. 4(a) and Fig. 5, we can observe that the flexural rigidity of the first story is reduced from 0.7 to 0.5 while those of the other stories remain around 1.0. This result implies that we can successfully update the structural parameters from initial state to DS1 and then to DS2 repeatedly. Although the prior model may provide outdated information of structural parameters, structural changes can be updated using a new set of measurement data based on our updating method. Cases I, II, and III yield $\xi$ values of 1.35%, 1.10%, and 0.64%, respectively, while Cases IV and V yield relatively larger $\xi$ values of 4.61% and 6.42%, respectively. The first three cases yield more accurate results than the other cases as similar to the DS1 scenario. The average $\xi$ value is 2.82%.

2) 60% partial measurement case (Three DOFs)

When 60% of measurements are available (three out of five DOFs), there are ten combinations of sensor locations as shown in Table V. The same damage scenarios (DS1, DS2) are used. For DS1, sensor location Cases I, II, and III yield $\xi$ values of 0.66%, 0.74%, and 0.43%, respectively. All these cases involve acceleration measurements from the first and second stories (1st and 2nd DOFs), which have large sensitivities to the damage in the first story member, as shown in Table III. In the other cases, the estimation errors range from 4% to 8%. The average $\xi$ value is 3.95%, which is slightly larger than that of the 80% partial measure case (2.48%) due to more severely lacking data. From Figs. 6(a) and 6(b), it is observed that the structural parameters are well estimated even when the measurement error is around 5% and only 60% of DOFs are measured.

Figs. 6(c) and 6(d) show the parameter estimation results for DS2. For sensor location Cases I through VIII, the normalized flexural rigidities of the damaged member (3rd story) are estimated to be about 0.8, whereas those of the undamaged members (1st, 2nd, 4th and 5th story) range from 0.9 to 1.1. In Case IX and X, the flexural rigidities of the damaged member are estimated about 0.9 while those of the 2nd story member are also estimated about 0.9. Thus, it is hard to distinguish the member that has changed (damaged) from others. This is because Cases IX and X include the 4th and 5th DOFs, which have the lowest sensitivities with respect to the 3rd story member, as shown in Table III.
3) 40% partial measurement case (Two DOFs)

When only 40% of DOFs are available for analysis (two out of five DOFs), ten combinations of sensor locations are available as shown in Table VI. The flexural rigidities and the estimation errors are obtained using the same procedure as the previous two cases. In DS1, since Case I includes 1st and 2nd DOFs that have large sensitivities with respect to the 1st story member, it results in the smallest estimation error (0.61%). On the other hand, Case X includes least sensitive DOFs and results in the largest estimation error (9.62%). Figs. 7(a) and 7(b) show that the structural parameters are well estimated in general. The average of \( \overline{\varepsilon} \) is 5.37%. The average \( \overline{\varepsilon} \) is slightly larger than that of 80% and 60% partial measurement cases. Based on these average \( \overline{\varepsilon} \) values for different levels of partial measurements (2.48%, 4.61%, and 5.37%, respectively), we can conclude that as the number of the available DOFs for measurements decreases, the structural parameter estimation becomes less accurate.

For DS2, the overall identification results are similar to those of the 60% partial measurement case, as shown in Figs. 7(c) and 7(d). However, the average \( \overline{\varepsilon} \) value is increased to 11.6%, and it is hard to distinguish which members are damaged in all the sensor location cases except Cases III and IV. In this damage scenario, we may need measurements from more than three DOFs for accurate parameter updating.

![Table V. Estimation Errors for 30% Damage with 60% Partial Measurements](image)

![Figure 7. Estimated flexural rigidities for various sensor location cases for 30% damage with 40% partial measurements](image)
TABLE VI. ESTIMATION ERRORS FOR 30% DAMAGE WITH 40% PARTIAL MEASUREMENTS

<table>
<thead>
<tr>
<th>Case</th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
<th>Case IV</th>
<th>Case V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured DOF</td>
<td>1st, 2nd</td>
<td>1st, 2nd</td>
<td>1st, 3rd</td>
<td>1st, 3rd</td>
<td>2nd, 3rd</td>
</tr>
<tr>
<td>Damage Scenario</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
</tr>
<tr>
<td>r</td>
<td>0.0061</td>
<td>0.1245</td>
<td>0.0209</td>
<td>0.1127</td>
<td>0.0673</td>
</tr>
<tr>
<td>Case</td>
<td>Case VI</td>
<td>Case VII</td>
<td>Case VIII</td>
<td>Case IX</td>
<td>Case X</td>
</tr>
<tr>
<td>Measured DOF</td>
<td>2nd, 4th</td>
<td>2nd, 4th</td>
<td>3rd, 4th</td>
<td>3rd, 5th</td>
<td>4th, 5th</td>
</tr>
<tr>
<td>Damage Scenario</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
<td>DS1, DS2</td>
</tr>
<tr>
<td>r</td>
<td>0.0596</td>
<td>0.1126</td>
<td>0.0608</td>
<td>0.1138</td>
<td>0.0589</td>
</tr>
</tbody>
</table>

V. CONCLUSION

A new methodology to estimate element-wise structural parameters using partial measurements is introduced based on a subspace SI framework. At first, subspace SI identifies system matrices of a state space model, which represents structural dynamics. To estimate structural parameters, such as stiffness and damping matrices, it is necessary to transform these system matrices in an arbitrary space to a physical space. This problem is defined as finding a similarity transformation matrix, which requires measurements from all DOFs. In practice, due to limitations of instrumentation, costs, and/or computation, it is difficult to obtain measurements from a complete set of DOFs. When only partial data are available, finding the similarity transformation becomes a rank deficient problem. To overcome the rank deficiency, we imposed additional constraints using prior information, such as an existing FE model or known design parameters that provide stiffness and damping matrices, and updated it with measured data. In essence, we search for a similarity transformation matrix that minimizes the difference between the system equations derived from prior information and those from measurement data. This new method to compute the similarity transformation matrix with partial measurement is developed for both displacement and acceleration measurement cases.

To validate the method, numerical case studies are investigated using a five-story shear building model with varying sensor locations, amount of data available, and accuracy of prior information. We assumed the stiffness and damping matrices at initial intact state are available from a prior model and then updated the structural parameters using a new set of data, after damage has been introduced. Once the structural parameters are estimated, they can be used as prior information for the next round of updating. In spite of measurement and modeling errors (i.e., inaccuracy of prior information), the proposed method reliably identified stiffness matrix and flexural rigidities. As the number of measured DOFs decreased, the estimation accuracy decreased. If we choose the DOFs which have large sensitivities with respect to the damaged member, we can obtain improved identification results using the same number of sensors.

For further study, experimental verification will be performed and a method to combine multiple partial measurement based parameters estimations will be studied. This study will enhance the practicality of subspace SI for detailed element level structural health monitoring, such as damage localization and quantification.

REFERENCES


